Les cahiers
Leibniz

Recent results on well-balanced orientations

Attila Bernáth, Satoru Iwata, Tamás Király, Zoltán Király,
Zoltán Szigeti

Laboratoire G-SCOP
46 av. Félix Viallet, 38000 GRENOBLE, France
ISSN : 1298-020X

n^{o} 162

February 2008

Site internet : http://www.g-scop.inpg.fr/CahiersLeibniz/

Recent results on well-balanced orientations

Attila Bernáth* Satoru Iwata† Tamás Király‡ Zoltán Király§
Zoltán Szigeti¶

February 8, 2008

Abstract

In this paper we consider problems related to Nash-Williams' Strong Orientation Theorem and Odd-Vertex Pairing Theorem. These theorems date to 1960 and up to now not much is known about their relationship to other subjects in graph theory. We investigated many approaches to find a more transparent proof for these theorems and possibly generalizations of them. In many cases we found negative answers: counter-examples and NP-completeness results. For example we show that the weighted and the degree-constrained versions of the well-balanced orientation problem are NP-hard. We also show that it is NP-hard to find a minimum cost feasible odd-vertex pairing or to decide whether two graphs with some common edges have simultaneous well-balanced orientations or not.

Nash-Williams' original approach was to define best-balanced orientations with feasible odd-vertex pairings: we show here that not every best-balanced orientation can be obtained this way. However we prove that in the global case this is true: every smooth k-arc-connected orientation can be obtained through a k-feasible odd-vertex pairing.

The aim of this paper is to help to find a transparent proof for the Strong Orientation Theorem. In order to achieve this we propose some other approaches and raise some open questions, too.

Keywords: well-balanced orientation, odd-vertex pairing.

1 Introduction

In 1960 Nash-Williams proved his Strong Orientation Theorem about the existence of *well-balanced* (and *best-balanced*) orientations. In fact, he proved a stronger result, the so-called *Odd-Vertex Pairing Theorem*. There are many intriguing questions related to these two theorems, some of which are answered in this paper. For example we show that it is NP-hard to find a minimum cost well-balanced orientation (given the cost for the two possible orientations of each edge) or a well-balanced orientation satisfying lower and upper bounds on the out-degrees at each vertex. Analogous results are given for best-balanced orientations. We also prove that it is NP-hard to find a minimum cost feasible odd-vertex pairing (where the cost of choosing a pair of odd-degree vertices is given for each pair). We examine several properties of k-arc-connected orientations and in most of the cases we show by counter-examples that these do not extend to well-balanced orientations. Many of the results presented in this paper (although not all of them) already appeared in two technical reports [16, 2], in some cases we omit details and refer the reader to those reports. In order to make the paper easier to read, the presentation of our results begins with the most natural and straightforward questions and then moves on to the more involved and sophisticated topics.

*Corresponding author. Dept. of Operations Research, Eötvös University, Pázmány P. s. 1/C, Budapest, Hungary H-1117. Research supported by MTA-ELTE Egerváry Research Group, OTKA grant K 060802 and the European MCRTN Adonet, Contract Grant No. 504438. E-mail: `bernath@cs.elte.hu`

†Research Institute for Mathematical Sciences, Kyoto University, Kyoto 606-8502, Japan. A part of this research was done while this author was at University of Tokyo. E-mail: `iwata@kurims.kyoto-u.ac.jp`

‡Dept. of Operations Research, Eötvös University, Pázmány P. s. 1/C, Budapest, Hungary H-1117. Research supported by MTA-ELTE Egerváry Research Group, OTKA grants K 060802 and NK 67867 and the European MCRTN ADONET, Contract Grant No. 504438. E-mail: `tkiraly@cs.elte.hu`

§Dept. of Computer Science and Communication Networks Laboratory, Eötvös University, Pázmány P. s. 1/C, Budapest, Hungary H-1117. Research supported by MTA-ELTE Egerváry Research Group and OTKA grants K 060802, NK 67867 and T 046234. E-mail: `kiraly@cs.elte.hu`

¶G-SCOP, INPGrenoble, UJF, CNRS, 46 avenue Félix Viallet, 38031 Grenoble, France. This research was partially done when this author visited University of Tokyo supported by the 21 Century COE Program from the Ministry of Education, Culture, Sports, Science and Technology of Japan. E-mail: `Zoltan.Szigeti@g-scop.inpg.fr`

Let us give a more detailed overview of the results of this paper. Let $G = (V, E)$ be an undirected (or a directed) graph. For two vertices $u, v \in V$ of G the **local edge-connectivity** (**local arc-connectivity**) $\lambda_G(u, v)$ from u to v in G is defined to be the maximum number of pairwise edge (arc resp.) disjoint paths from u to v in G. The **global edge-connectivity** (**global arc-connectivity**) of a graph (digraph) G is $\min\{\lambda_G(u, v) : u, v \in V\}$. G is **k-edge-connected** (**k-arc-connected** resp.) if $\lambda_G(u, v) \geq k$ for every $(u, v) \in V \times V$ (i.e. its global edge- (arc-) connectivity is at least k). More generally, for $U \subseteq V$, G is **k-edge-connected** (**k-arc-connected** resp.) **in U** if $\lambda_G(u, v) \geq k$ for every $(u, v) \in U \times U$.

Nash-Williams' Strong Orientation Theorem [22] states that for any undirected graph G there exists an orientation $\vec{G}$ of G for which $\lambda_{\vec{G}}(u, v) \geq \lfloor \frac{1}{2}\lambda_G(u, v)\rfloor$ for every $(u, v) \in V \times V$. An orientation with this property will be called **well-balanced**. For global edge-connectivity this specializes to the following Weak Orientation Theorem: G has a k-arc-connected orientation if and only if G is $2k$-edge-connected. In this paper we will always refer to the *global case* when we want to specialize a question to global edge- (or arc-) connectivity.

Let $G = (V + s, E)$ be an undirected graph. The operation **splitting off** is defined as follows: two edges rs, st incident to s are replaced by a new edge rt. The splitting-off theorem of Lovász [18] concerns global edge-connectivity: if G is k-edge-connected in V and $d(s)$ is even, then there exists a pair of edges rs, st incident to s whose splitting off maintains the k-edge-connectivity in V, where $k \geq 2$. Lovász [18] also showed that the Weak Orientation Theorem is an easy consequence of his splitting-off theorem. Mader [20] generalized Lovász' result to local edge-connectivity: if $d(s) \geq 4$ and no cut-edge of G is incident to s, then there exists a pair of edges rs, st incident to s whose splitting off maintains the local edge-connectivities in V. A simple proof for Mader's theorem can be found in [9]. Mader [20] provided a new proof for the Strong Orientation Theorem by applying his splitting-off theorem.

Let $\vec{G} = (V + s, A)$ be a directed graph. Splitting off can be naturally reformulated for directed graphs: two arcs rs, st are replaced by rt. Mader [21] proved a splitting-off theorem preserving global arc-connectivity in directed graphs: if G is k-arc-connected in V and $\varrho(s) = \delta(s)$ then there exists a pair of arcs rs, st incident to s whose splitting off maintains the k-arc-connectivity in V. An example of Enni [6] shows that there is no splitting-off theorem preserving local arc-connectivities in directed graphs. In Question 2 we provide a smaller example showing that even if $\vec{G}$ is a well-balanced orientation of G there is no splitting off that preserves local arc-connectivities in V.

Nash-Williams' Odd-Vertex Pairing Theorem [22] states that every undirected graph G has a **pairing** M (a set of new edges on the set T_G of odd degree vertices of G such that $d_M(v) = 1$ for every $v \in T_G$) that is **feasible** ($d_M(X) \leq b_G(X)$ for every $X \subseteq V$, where $b_G(X)$ is the number of "extra" edges leaving X, for the formal definition see the next session). A simpler proof of the Odd-Vertex Pairing Theorem can be found in [10]. For the global case, let us call a pairing M to be **k-feasible** (where k is a nonnegative integer) if $d_M(X) \leq d_G(X) - 2k$ for every $\emptyset \neq X \subsetneq V$. It was shown in [17] that in this case the Odd-Vertex Pairing Theorem (i.e. the existence of a k-feasible pairing in a $2k$-edge-connected graph) can be proven easily by the global splitting-off theorem.

The Strong Orientation Theorem is trivial for Eulerian graphs (any Eulerian orientation will do), nevertheless this special case plays an important role in the theory. It was shown in [17] that for Eulerian graphs, an orientation is well-balanced if and only if it is Eulerian.

Nash-Williams [22] showed that if M is a feasible pairing of G then for every Eulerian orientation $\vec{G}+\vec{M}$ of $G+M$, $\vec{G}$ is well-balanced and furthermore it is **smooth**, that is the in-degree and the out-degree of every vertex differ by at most one. A smooth well-balanced orientation is called **best-balanced**. A related result in [17] states that for each (not necessarily feasible) pairing M of G there exists an Eulerian orientation $\vec{G} + \vec{M}$ of $G + M$ so that $\vec{G}$ is best-balanced. We show (Question 7) that not every best-balanced orientation can be obtained from a feasible pairing this way. On the other hand we prove in Theorem 8.2 that in the global case every smooth k-arc-connected orientation can be obtained from some k-feasible pairing using this construction.

The above mentioned two proofs of the odd-vertex pairing theorem (the original due to Nash-Williams and that of Frank) both imply a polynomial algorithm to find a feasible odd-vertex pairing, though it is not explicitly stated in either of them. An explicit algorithm for this problem is sketched in [13], where it is stated that an odd-vertex pairing (and consequently a best-balanced orientation) can be found in $O(nm^2)$ time in a graph and in $O(n^6)$ time in a multigraph. It is a natural question to look for a feasible odd-vertex pairing of minimum cost where the cost for any pair of odd-degree vertices is given. However we show (Corollary 9.2) that this problem is NP-complete, even for the global case. Another natural question is whether one can find a well-balanced orientation of minimum cost (with costs given for the two possible orientations of every edge) or whether one can find a well-balanced orientation satisfying some other constraints, for example lower and upper bounds on the out-degrees at each vertex. In his survey paper [10] Frank mentions these questions when he writes the following about his proof of the odd-vertex pairing theorem: *I keep feeling that there must be an even more illuminating proof which finally will lead to methods to solve the minimum cost and/or degree-constrained well-balanced orientation problem.* Here we present negative answers in this direction: we prove the NP-completeness of these problems (see Theorem 4.3). We have similar results for best-balanced orientations.

Nash-Williams [23] formulated the following extension of the Strong Orientation Theorem for a subgraph chain of length two: if H is a subgraph of G, then there exists a best-balanced orientation of H that can be extended to a best-balanced orientation of G. A simple proof is given in [17]: it is shown there that the Odd-Vertex Pairing Theorem easily implies this, and that the global case of this extension has a simple proof. We show that the *general subgraph-chain property* is not valid, that is, this extension cannot be generalized for a subgraph chain of length three, not even in the case of global edge-connectivity (see Question 5).

The authors of [17] generalized further the above extension by showing that the following *edge disjoint subgraphs property* is valid: if $\{G_1, G_2, ..., G_k\}$ is a partition of G into edge disjoint subgraphs then there is an orientation $\vec{G}$ of G such that each $\vec{G}_i$ and $\vec{G}$ are best-balanced orientations of G_i and of G. We show that deciding for two non-edge-disjoint graphs whether they have simultaneous best-balanced orientations is NP-complete, even for two Eulerian graphs (see Question 6).

Both the original proof of the Odd-Vertex Pairing Theorem in [22] and Frank's proof [10] rely heavily on the skew-submodularity of the set function b_G. We show (Question 8) that the existence of a feasible pairing cannot be generalized to arbitrary skew-submodular functions. Skew-submodular functions correspond to local edge-connectivity, while crossing submodular functions can be considered as generalizations of global edge-connectivity. For such a function it is an open problem whether there exists a feasible pairing. However the corresponding orientation theorem can be proved easily (see Theorem 10.1).

Frank [8] proved the following *reorientation property* for k-arc-connected orientations: given two k-arc-connected orientations of G, there exists a series of k-arc-connected orientations of G (leading from the first to the second given orientation), such that in each step we reverse a directed path or a circuit. For well-balanced (or best-balanced) orientations it is not known whether the reorientation property is valid.

The proof of Frank in [8] easily implies the following *matroid property* for smooth k-arc-connected orientations: the family of sets, over smooth k-arc-connected orientations, consisting of vertices whose in-degree is larger than the out-degree, forms the family of bases of a matroid. We show that this is not true in general for best-balanced orientations (see Question 12).

Frank [7] also proved that the *linkage property* is valid for the k-arc-connected orientation problem, i.e. there exists a k-arc-connected orientation whose in-degree function satisfies lower and upper bounds if and only if there is one satisfying the lower bound and one satisfying the upper bound. É. Tardos [24] showed that the linkage property is not valid for the well-balanced orientation problem. Here we present another example (see Question 14).

The rest of the paper is organized as follows. In Section 2 we introduce some further notations. In Section 3 we summarize known results on well-balanced orientations and odd-vertex pairings. In Section 4 we consider well-balanced orientations with extra requirements: we prove the NP-completeness of questions such as finding a well-balanced orientation of minimum cost or one satisfying lower and upper bounds on the out-degrees. In Section 5 we consider mixed graphs and their well-balanced orientations. Section 6 is devoted to the splitting-off operation. In Section 7 we consider the question of orienting several graphs with possibly some common edges resulting in an orientation that is simultaneously well-balanced. Section 8 asks whether every best-balanced orientation can be obtained from a feasible odd-vertex pairing. In the next section we investigate the structure of feasible pairings. In Section 10 we introduce a more general setting and investigate feasible pairings for connectivity functions. In the last section we show that the matroid property which is valid for k-arc-connected orientations, does not extend to well-balanced orientations.

2 Notation

A directed graph is denoted by $\vec{G} = (V, A)$ and an undirected graph by $G = (V, E)$. For a directed graph $\vec{G}$ and a set $X \subseteq V$ let $\delta_{\vec{G}}(X) := |\{uv \in A : u \in X,\ v \notin X\}|$ (the *out-degree of X in $\vec{G}$*), $\varrho_{\vec{G}}(X) := \delta_{\vec{G}}(V - X)$ (the *in-degree of X in $\vec{G}$*), and $f_{\vec{G}}(X) := \varrho_{\vec{G}}(X) - \delta_{\vec{G}}(X)$. If $z : A \to \mathbb{R}$ then let $\delta^z_{\vec{G}}(X) := \sum_{\{uv \in A:\ u \in X,\ v \notin X\}} z(uv)$ and $\varrho^z_{\vec{G}}(X) := \delta^z_{\vec{G}}(V - X)$. For a digraph $\vec{G}$ and $u, v \in V$ let

$\lambda_{\vec{G}}(u,v) := \min\{\delta_{\vec{G}}(Y) : Y \subseteq V,\ u \in Y,\ v \notin Y\}$ (by Menger's theorem this is indeed an equivalent definition of the *local arc-connectivity from* u *to* v *in* $\vec{G}$) and $\overleftarrow{G} := (V, \{vu : uv \in A\})$. Observe that $\forall X \subseteq V$

$$f_{\vec{G}}(X) \quad = \quad \sum_{v \in X} f_{\vec{G}}(v). \tag{1}$$

For an undirected graph G and a set $X \subseteq V$ let $d_G(X) := |\{uv \in E : u \in X,\ v \notin X\}|$ (the *degree of* X *in* G) and $i_G(X) := |\{uv \in E : u, v \in X\}|$ (the *number of edges induced by* X). For two sets $X, Y \subseteq V$ let $d_G(X,Y) := |\{uv \in E : u \in X - Y,\ v \in Y - X\}|$. If $u, v \in V$, then let $\lambda_G(u,v) := \min\{d_G(Y) : Y \subseteq V,\ u \in Y,\ v \notin Y\}$, (again by Menger's theorem this is indeed an equivalent definition of the *local edge-connectivity between* u *and* v *in* G). Introduce the set functions $R_G(X) := \max\{\lambda_G(x,y) : x \in X,\ y \notin X\}$ (with $R_G(\emptyset) = R_G(V) = 0$), $\hat{R}_G(X) := 2\lfloor R_G(X)/2 \rfloor$, $b_G(X) := d_G(X) - \hat{R}_G(X)$ and let $T_G := \{v \in V : d_G(v) \text{ is odd}\}$. The undirected graph $G = (V, E)$ is **connected** if for every pair of vertices u, v there is a (u,v)-path in G. It is k**-edge-connected** if $G - F$ is connected for any $F \subseteq E$ with $|F| \leq k - 1$. For a function $\mathbf{r} : V \times V \to \mathbb{Z}_0^+$, we say that G is **r-edge-connected** if $\lambda_G(u,v) \geq \mathbf{r}(u,v)$ for every pair u, v of vertices.

The directed graph $\vec{G} = (V, A)$ is **strongly connected** if for every ordered pair $(u,v) \in V \times V$ of vertices there is a directed (u,v)-path in $\vec{G}$. It is k**-arc-connected** if $\vec{G} - F$ is strongly connected for any $F \subseteq A$ with $|F| \leq k - 1$. For a function $\mathbf{r} : V \times V \to \mathbb{Z}_0^+$, we say that $\vec{G}$ is **r-arc-connected** if $\lambda_{\vec{G}}(u,v) \geq \mathbf{r}(u,v)$ for every ordered pair u, v of vertices.

An orientation $\vec{G}$ of G is called **well-balanced** if $\vec{G}$ satisfies (2), **smooth** if $\vec{G}$ satisfies (3) and **best-balanced** if it is smooth and well-balanced. Note that if $\vec{G}$ is best-balanced then so is $\overleftarrow{G}$. Let us denote by $\mathcal{O}_w(G)$ and $\mathcal{O}_b(G)$ the set of well-balanced and best-balanced orientations of G.

$$\lambda_{\vec{G}}(x,y) \quad \geq \quad \lfloor \lambda_G(x,y)/2 \rfloor \ \ \forall\ (x,y) \in V \times V, \tag{2}$$
$$|f_{\vec{G}}(v)| \quad \leq \quad 1 \quad\ \forall\ v \in V. \tag{3}$$

A **pairing** M of G is a new graph on the set of odd-degree vertices T_G in which each vertex has degree one. Let M be a pairing of G. An orientation $\vec{M}$ of M that satisfies (4) is called **good**. Note that, by Claim 3.5 of the next section, if $\vec{M}$ is good then every Eulerian orientation $\vec{G} + \vec{M}$ of $G + M$ that extends $\vec{M}$ defines a best-balanced orientation of G. Pairing M is **well-orientable** if there *exists* a good orientation of M, M is **strong** if *every* orientation of M is good and M is **feasible** if (5) is satisfied. Clearly an oriented pairing $\vec{M}$ is good iff $\overleftarrow{M}$ is good. Let us denote by $\mathcal{P}_f(G)$ the set of feasible pairings of G.

$$f_{\vec{M}}(X) \quad \leq \quad b_G(X) \quad \forall X \subseteq V, \tag{4}$$
$$d_M(X) \quad \leq \quad b_G(X) \quad \forall X \subseteq V. \tag{5}$$

We shall use the facts that for an undirected graph G and subsets $X, Y, Z \subseteq V$ we have

$$d_G(X) + d_G(Y) \quad = \quad d_G(X \cap Y) + d_G(X \cup Y) + 2d_G(X,Y), \tag{6}$$
$$\begin{aligned} d_G(X) + d_G(Y) + d_G(Z) \quad \geq \quad & d_G(X \cap Y \cap Z) + d_G(X - (Y \cup Z)) + \\ & d_G(Y - (X \cup Z)) + d_G(Z - (X \cup Y)). \end{aligned} \tag{7}$$

3 Known results

The following four theorems are due to Nash-Williams [22, 23]. First we state the **Odd-Vertex Pairing Theorem**: this theorem is particularly interesting since it has practically no known connection to any other result in graph theory.

Theorem 3.1 (**Odd-Vertex Pairing Theorem**). *Every graph has a feasible pairing.*

The Odd-Vertex Pairing Theorem easily implies the following **Strong Orientation Theorem**.

Theorem 3.2 (**Strong Orientation Theorem**). *Every graph has a best-balanced orientation.*

In fact, the Odd-Vertex Pairing Theorem also implies the following, stronger result (for a proof see [17]).

Theorem 3.3. *For every subgraph H of G, there exists a best-balanced orientation of H that can be extended to a best-balanced orientation of G.*

A simple consequence of the Strong Orientation Theorem is the **Weak Orientation Theorem** that concerns global edge-connectivity instead of local edge-connectivity. While the only proof of the Strong Orientation Theorem is via the Odd-Vertex Pairing Theorem, for the Weak Orientation Theorem we know different proof methods, polyhedral generalizations and we understand much better the relationships with submodular functions and polymatroids.

Theorem 3.4 (**Weak Orientation Theorem**)**.** *A graph G has a k-arc-connected orientation if and only if G is $2k$-edge-connected.*

In [17] Király and Szigeti proved the following results.

Claim 3.5. *The following statements are equivalent:*

$$\vec{G} \in \mathcal{O}_w(G), \tag{8}$$

$$\delta_{\vec{G}}(X) \geq \lfloor \frac{R(X)}{2} \rfloor \qquad \forall\, X \subseteq V, \tag{9}$$

$$f_{\vec{G}}(X) \leq b_G(X) \qquad \forall\, X \subseteq V. \tag{10}$$

Claim 3.6. *A pairing is feasible if and only if it is strong.*

Theorem 3.7. *Every pairing is well-orientable.*

Using these observations they also proved in [17] the following two generalizations of Theorem 3.2.

Theorem 3.8. *For every partition $\{E_1, E_2, ..., E_k\}$ of $E(G)$, if $G_i = (V, E_i)$ then G has a best-balanced orientation $\vec{G}$, such that the inherited orientation of each G_i is also best-balanced.*

Theorem 3.9. *For every partition $\{X_1, ..., X_l\}$ of $V = V(G)$, G has an orientation $\vec{G}$ such that $\vec{G}$, $((\vec{G}/X_1)...)/X_l$ and $\vec{G}/(V - X_i)$ $(1 \leq i \leq l)$ are best-balanced orientations of the corresponding graphs.*

4 Well-balanced orientations with extra requirements

It is a natural question whether one can find a well-balanced orientation of minimum cost (with costs given for the two possible orientations of every edge) or whether one can find a well-balanced orientation satisfying some other constraints, for example lower and upper bounds on the out-degrees at each vertex. Here we present negative answers in this direction: we prove the NP-completeness of these problems. Let us introduce the problems we want to consider and give some motivation.
For well-balanced orientations we look at the following problems:

Problem 1. : MinCostWellBalanced
Instance: *A graph G, nonnegative integer costs for the two possible orientations of each edge, and an integer bound K.*
Question: *Is there a well-balanced orientation of G with total cost at most K?*

Problem 2. : BoundedWellBalanced
Instance: *A graph $G = (V, E)$, $l, u : V \to \mathbb{Z}_+$ bounds with $l \leq u$.*
Question: *Is there a well-balanced orientation $\vec{G}$ of G with $l(v) \leq \delta_{\vec{G}}(v) \leq u(v)$ for every $v \in V$?*

Problem 3. : MinVertexCostWellBalanced
Instance: *A graph G, integer costs $c : V \to \mathbb{Z}$, integer B.*
Question: *Is there a well-balanced orientation $\vec{G}$ of G with $\sum_{v \in V} c(v) \cdot \delta_{\vec{G}}(v) \leq B$?*

For best-balanced orientations we define problems MinCostBestBalanced, BoundedBestBalanced and MinVertexCostBestBalanced similarly, changing the phrase well-balanced to best-balanced.

Problems MinCostWellBalanced and MinCostBestBalanced are quite natural weighted versions of the original problem, the problem of finding a well-balanced or a best-balanced orientation. The constrained versions BoundedWellBalanced and BoundedBestBalanced also arise naturally: they are mentioned in the survey paper of András Frank [10] and a related problem, when we have only bounds from one side (say, upper bounds) in a best-balanced orientation is still an open problem mentioned in [5] (though we have to mention that a related question was shown to be hard, namely it has been shown by [1] that it is NP-hard to decide whether a graph has an **r**-arc-connected orientation with upper bounds on the out-degrees even for a $0-1$-valued symmetric function **r**). The third approach is motivated by the following observation: in an orientation problem with arc-connectivity requirements, finding the out-degree function of a solution is polynomially equivalent with finding a solution. The authors of [16] introduce the following polyhedron for a graph $G = (V, E)$ (see Section 9 in [16]):

$$\begin{aligned} P := \quad & \{x \in \mathbb{R}^V : x(Z) \geq i_G(Z) + \lfloor R_G(Z)/2 \rfloor \quad \forall Z \subseteq V,\ x(V) = |E|, \\ & \lfloor d_G(v)/2 \rfloor \leq x(v) \leq \lceil d_G(v)/2 \rceil \quad \forall v \in V\}. \end{aligned}$$

The integer hull of this polyhedron is the convex hull of the out-degree functions of all best-balanced orientations of G. However it is proved in [16] that this polyhedron is not necessarily integral: here we prove that optimization over the integer hull of this polyhedron (that is, problem MinVertexCostBestBalanced) is NP-hard. Problem MinVertexCostWellBalanced is just the counterpart of this problem for well-balanced orientations.

Now we give some known results that will be needed later. The following is a simple observation: the proof is left to the reader.

Lemma 4.1. *If $\vec{G}$ and $\vec{G}'$ are two orientations of a graph $G = (V, E)$ with $\delta_{\vec{G}}(x) = \delta_{\vec{G}'}(x)$ for all $x \in V$ then $\vec{G}'$ can be obtained from $\vec{G}$ by reversing directed cycles.* □

Corollary 4.2. *If $\vec{G}$ and $\vec{G}'$ are two orientations of a graph $G = (V, E)$ with $\delta_{\vec{G}}(x) = \delta_{\vec{G}'}(x)$ for all $x \in V$ then*

$$\vec{G} \text{ is well-balanced} \iff \vec{G}' \text{ is well-balanced.}$$

Proof. Directly from lemma 4.1. Alternatively, we can show that $\lambda_{\vec{G}}(x, y) = \lambda_{\vec{G}'}(x, y)$ for all $x, y \in V$ using the fact $\delta_{\vec{G}}(X) = \sum_{x \in X} \delta_{\vec{G}}(x) - i_G(X) = \delta_{\vec{G}'}(X)$ for any $X \subseteq V$. □

For well-balanced orientations we have the following results.

Theorem 4.3. *Problems* MinCostWellBalanced, BoundedWellBalanced *and* MinVertexCostWellBalanced *are NP-complete.*

Proof. The problems are clearly in NP. In order to show their completeness we will give a reduction from Vertex Cover (see [14], Problem GT1). For a given instance $G' = (V', E')$ and $k \in \mathbb{N}$ of the Vertex Cover problem (where we can assume that the minimum degree in G' is at least 2) consider the following undirected graph $G = (V, E)$. The vertex set V will contain one designated vertex s, $d_{G'}(v) + 1$ vertices $x^v_0, x^v_1, x^v_2, \ldots, x^v_{d_{G'}(v)}$ for every $v \in V'$, and one vertex x_e for every $e \in E'$. Let us fix an ordering of V', say $V' = \{v_1, v_2, \ldots, v_n\}$. The edge set E contains a circuit on $s, x^{v_1}_0, x^{v_2}_0 \ldots, x^{v_n}_0$ in this order, one edge from s to x^v_1 for every $v \in V'$, edges between x^v_i and x^v_{i+1} for every $v \in V'$ and every i between 0 and $d_{G'}(v) - 1$, two parallel edges between s and x_e for every $e \in E'$ and finally for each $v \in V'$ take an arbitrary order of the $d_{G'}(v) = d$ edges of G' incident to v, say $e^1, e^2, \ldots, e^d$ and include the edge $x^v_i x_{e^{i-1}}$ for any $2 \leq i \leq d - 1$ and the edges $x^v_d x_{e^{d-1}}$ and $x^v_d x_{e^d}$ (i.e. distribute the edges of G' incident to v arbitrarily among vertices $x^v_2, \ldots, x^v_d$ resulting $d_G(x^v_i) = 3$ for each $2 \leq i \leq d$).

The construction is illustrated in Figure 1. The edges drawn bold indicate a multiplicity of 2.

Notice that for every $v \in V'$ and $0 \leq i \leq d_{G'}(v)$ we have $d_G(x^v_i) = 3$ and for every $e \in E'$ we have $d_G(x_e) = 4$. What is more, it is easy to check, that $\lambda_G(x, y) = \min(d_G(x), d_G(y))$ for every $x, y \in V$ (for example one can check that this is true if $y = s$ from which it follows for arbitrary x, y).

Define a partial orientation of G: orient the circuit $s, x^{v_1}_0, x^{v_2}_0 \ldots, x^{v_n}_0$ to become a directed circuit in this order, orient the edges from x^v_i to x^v_{i+1} for every $v \in V'$ and every i between 0 and $d_{G'}(v) - 1$, orient the two parallel edges from x_e towards s for every $e \in E'$ and finally for each $v \in V'$, $2 \leq i \leq d_{G'}(v)$ and $e \in E'$ if there is an edge between x^v_i and x_e then orient this edge from x^v_i to x_e (so we have given the orientation of every edge except those of form sx^v_1 for $v \in V'$). Figure 2 is an illustration.

Let us denote the subgraph $G - \{sx^v_1 : v \in V'\}$ by G_1 and the orientation of this graph given above by $\vec{G}_1$. Observe that $\vec{G}_1$ is a strongly connected graph and $\lambda_{\vec{G}_1}(x_e, s) = 2$ for each $e \in E'$.

Claim 4.4. *Problem* MinCostWellBalanced *is NP-complete.*

Proof: For a given instance $G' = (V', E')$ and $k \in \mathbb{N}$ of Vertex Cover consider the following instance of MinCostWellBalanced: let the graph G be as described above, let $K = k$ be the bound on the total cost and define the orientation-costs as follows. Orienting the edges of G_1 as in $\vec{G}_1$ costs nothing,

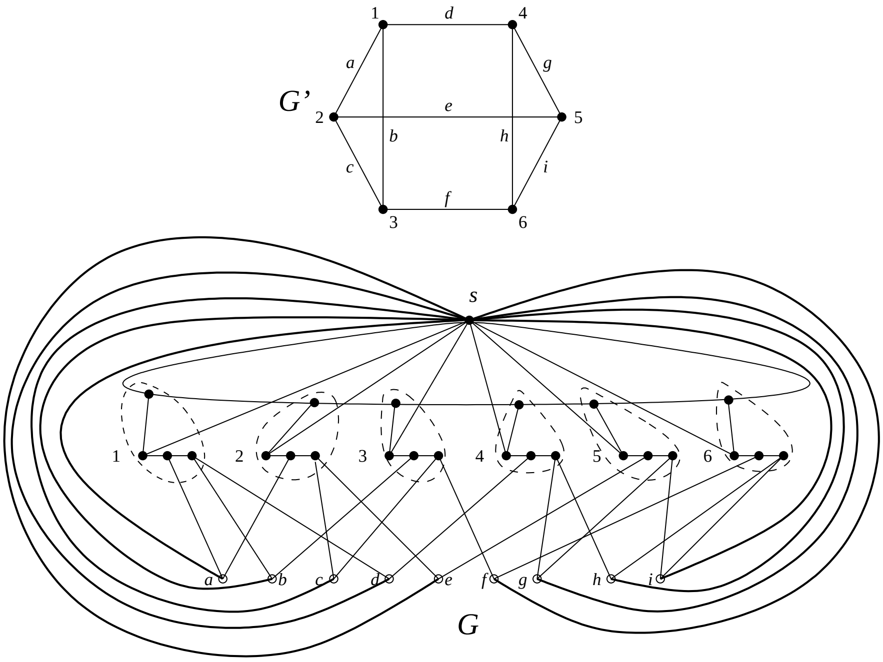

Figure 1: Construction of graph G

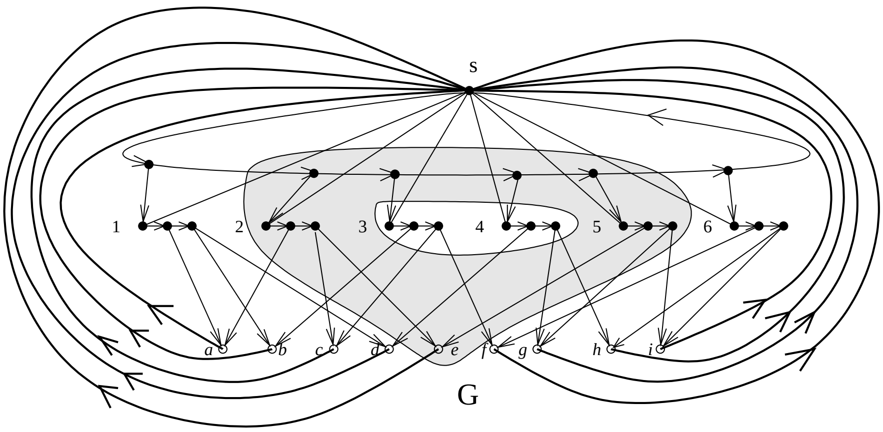

Figure 2: The partial orientation and the cut

but giving any edge the reverse orientation will cost $k+1$. It remains to define the costs of orientations of edges between s and x_1^v for each $v \in V'$: such an edge costs 1, if oriented from s to x_1^v and 0 in the other direction. So we only have freedom choosing the orientation of these edges, if we don't want to exceed the cost limit k.

First we claim that if there is a vertex cover $S \subseteq V'$ of size not more than k then there is a well-balanced orientation $\vec{G}$ of G of cost not more than k: for each $v \in S$ orient the edge sx_1^v from s to x_1^v and orient the other edges in the direction which costs nothing. This has clearly cost at most k and it is easy to check that $\lambda_{\vec{G}}(s, x_e) = 2$ for each $e \in E'$ which together with the former observations gives that $\vec{G}$ is well-balanced.

On the other hand suppose that we have found a well-balanced orientation $\vec{G}$ of G of cost at most k: this is possible only if there are at most k vertices in V' such that the edges sx_1^v are oriented from s to x_1^v exactly for these edges and all the other edges are oriented in the direction which costs 0. We claim that these vertices form a vertex cover of G': if the edge $e = v_j v_l \in E'$ was not covered (where $j < l$ are the indices of the vertices in the fixed ordering), then $\varrho_{\vec{G}}(X) = 1$ would contradict the well-balancedness of $\vec{G}$, where

$$X = \{x_e\} \bigcup \{x_0^{v_i} : j \le i \le l\}$$
$$\bigcup \{x_i^{v_j} : 1 \le i \le d_{G'}(v_j)\} \bigcup \{x_i^{v_l} : 1 \le i \le d_{G'}(v_l)\}$$

(Figure 2 illustrates the cut, too). □

Claim 4.5. *Problem* BoundedWellBalanced *is NP-complete.*

Proof: For an instance $G' = (V', E')$ and $k \in \mathbb{N}$ of Vertex Cover consider the following instance of BoundedWellBalanced: let the graph G be as described above and upper bound on the out-degree of s given by $u(s) = k + 1$, and lower bounds $l(x_i^v) = 2$ for each $v \in V'$ and $i \in \{0, 2, 3, \ldots, d_{G'}(v)\}$ (observe that these are in fact exact prescriptions for these out-degrees, notice, that we excluded $i = 1$): the other bounds can be trivial, that is $l(x) = 0$ and $u(x) = d_G(x)$ if it was not specified otherwise. We refer the reader to [2] for the details. □

Claim 4.6. *Problem* MinVertexCostWellBalanced *is NP-complete.*

Proof: For an instance $G' = (V', E')$ and $k \in \mathbb{N}$ of Vertex Cover consider the following instance of MinVertexCostWellBalanced: let the graph G be as described above and vertex-costs the following: let $c(s) = 1$ and $c(x_i^v) = -k$ for each $v \in V'$ and $i \in \{0, 2, 3, \ldots, d_{G'}(v)\}$ (and zero for the rest of the vertices). Finally, let $B = -4k|E'| + k + 1$. For more details see [2]. □

For best-balanced orientations we have the following corresponding results.

Theorem 4.7. *Problems* MinCostBestBalanced, BoundedBestBalanced *and* MinVertexCost-BestBalanced *are NP-complete.*

Proof. The problems are clearly in NP. To show completeness we give a reduction from Vertex Cover as before, but we need to change the construction a bit. For a given instance $G' = (V', E')$ and $k \in \mathbb{N}$ of the Vertex Cover problem (where $d_{G'}(v) \ge 2$ is again assumed for any $v \in V'$), modify the construction of the graph $G = (V, E)$ as follows: add $2|E'| + |V'| - 2k = N$ new vertices $z_1, z_2, \ldots, z_N$ and connect each of these vertices with s. So these new vertices will have degree 1 and s will have degree $4|E'| + 2|V'| + 2 - 2k$ in G. Denote this modified graph with $G = (V, E)$.

Define again a partial orientation of G: this is the same as the one defined above in the first construction, with the addition that for each i between 1 and N orient the edge sz_i from s to z_i.

Again call the subgraph $G - \{sx_1^v : v \in V'\}$ by G_1 and the above given orientation of this graph by $\vec{G}_1$. Again we have $\lambda_G(x, y) = \min(d_G(x), d_G(y))$ for every $x, y \in V$, $\lambda_{\vec{G}_1}(x, y) \ge 1$ for every $x, y \in V - \{z_1, z_2, \ldots, z_N\}$ and $\lambda_{\vec{G}_1}(x_e, s) = 2$ for each $e \in E'$.

Claim 4.8. *Problem* MinCostBestBalanced *is NP-complete, even for* 0–1 *orientation costs.*

Proof: For a given instance $G' = (V', E')$ and $k \in \mathbb{N}$ of Vertex Cover consider the following instance of MinCostBestBalanced: let the graph G be as described above, let $K = 0$ be the bound on the total cost and define the orientation-costs as follows. Orienting the edges of G_1 as in $\vec{G}_1$ costs nothing, but giving any edge the reverse orientation will cost 1. It remains to define the costs of orientations of edges between s and x_1^v for each $v \in V'$: these edges can be oriented in any direction with 0 cost. Details again can be found in [2]. □

Claim 4.9. *Problem* BoundedBestBalanced *is NP-complete.*

Proof: For an instance $G' = (V', E')$ and $k \in \mathbb{N}$ of Vertex Cover consider the following instance of BoundedBestBalanced: let the graph G be as described above and bounds on the out-degrees of odd degree vertices of G given as follows (of course, for even-degree vertices $x \in V$ one has $l(x) = d_G(x)/2 = u(x)$):

- $l(x_i^v) = 2 = u(x_i^v)$ for each $v \in V'$ and $i \in \{0, 2, 3, \ldots, d_{G'}(v)\}$ (exact prescriptions),
- $l(z_i) = 0 = u(z_i)$ for each $i = 1, 2, \ldots, N$ (exact prescriptions),
- $l(x_1^v) = 1$ and $u(x_1^v) = 2$ for each $v \in V'$ (so we only have freedom here).

For the details see [2]. □

Claim 4.10. *Problem* MinVertexCostBestBalanced *is NP-complete.*

Proof: For an instance $G' = (V', E')$ and $k \in \mathbb{N}$ of Vertex Cover consider the following instance of MinVertexCostWellBalanced: let the graph G be as described above and vertex-costs the following: let $c(z_i) = 1$ for each $i = 1, 2, \ldots, N$ and $c(x_i^v) = -1$ for each $v \in V'$ and $i \in \{0, 2, 3, \ldots, d_{G'}(v)\}$ (and zero for the rest of the vertices). Finally, let $B = -2\left(\sum(d_{G'}(v) : v \in V')\right) = -4|E'|$. Details again can be found in [2]. $\square$

5 Mixed graphs

A *mixed graph* is determined by the triple (V, E, A) where V is the set of vertices, E is the set of undirected edges and A is the set of directed arcs. The *underlying undirected graph* is obtained by deleting the orientation of the arcs in A. An *orientation of a mixed graph* means that we orient the undirected edges (and keep the orientation of directed arcs).

A possible way to prove the Strong Orientation Theorem could be to characterize mixed graphs whose undirected edges can be oriented to have a well-balanced orientation of the underlying undirected graph. The following problem was mentioned in Section 4.2 of [16]:

Problem 4. *Given a mixed graph, decide whether it has an orientation that is a well-balanced orientation of the underlying undirected graph.*

At the time of the submission of the present paper the status of this problem was unknown, but during the revision process it was shown to be NP-complete in [3]. However, the proof of Claim 4.8 immediately gives the NP-completeness of the following, related problem.

Problem 5. *Given a mixed graph, decide whether it has an orientation that is a best-balanced orientation of the underlying undirected graph.*

We have to mention that the global edge-connectivity version of these questions can be solved: one can decide whether a mixed graph has a k-arc-connected orientation, even with the presence of lower and upper bounds on the out-degrees of the required orientation, see [12].

6 Splitting off

For an undirected graph $G = (V+s, E)$ let the graph obtained by splitting-off the edges $rs, st \in E$ be denoted by G_{rt}, i.e. $G_{rt} := G - \{rs, st\} + rt$. Similarly, for a directed graph $\vec{G} = (V+s, A)$ with $rs, st \in A$ let $\vec{G}_{rt} = \vec{G} - \{rs, st\} + rt$. Alternatively, $\vec{G}_{rt}$ can also mean an orientation of G_{rt} with $rt \in A(\vec{G}_{rt})$.

We have seen in the introduction that splitting-off theorems are very useful in the proof of the Strong and the Weak Orientation Theorem. We also mention that Mader's proof [20] for the Strong Orientation Theorem as well as Frank's proof [10] for Theorem 3.1 uses Mader's splitting-off theorem.

The Odd-Vertex Pairing Theorem would be an easy task if the following was true.

Question 1. *For every 2-edge-connected graph G there exists a pair of incident edges rs, st such that*

$$b_G(X) \geq b_{G_{rt}}(X) \quad \forall X \subseteq V. \tag{11}$$

Counter-example 1. Let $G = (U, V; E)$ be the complete bipartite graph $K_{3,4}$. Let us denote the vertices as follows: $U := \{a, b, c, d\}$ and $V := \{x, y, z\}$. By symmetry, $\{rs, st\}$ is either $\{xd, dy\}$ or $\{az, zb\}$. In the first case $b_G(z) = 0 < 2 = b_{G_{xy}}(z)$ and in the second case $b_G(\{a, x, y\}) = 3 < 5 = b_{G_{ab}}(\{a, x, y\})$. In both cases (11) is violated. $\square$

Question 2. *If $\vec{G}$ is a best-balanced orientation of $G = (V+s, E)$ and $\varrho_{\vec{G}}(s) = \delta_{\vec{G}}(s)$ then there exist $rs, st \in A(\vec{G})$ so that*

$$\lambda_{\vec{G}_{rt}}(x, y) \geq \lambda_{\vec{G}}(x, y) \quad \forall (x, y) \in V \times V. \tag{12}$$

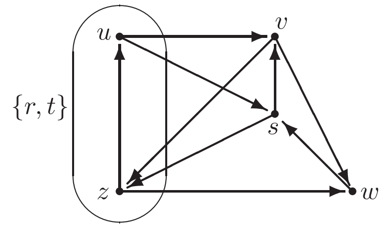

Figure 3: There is no good directed splitting off at s

Counter-example 2. Let $G=(V+s,E)$ and $\vec{G}=(V+s,A)$ be defined as follows (see Figure 3): $V:=\{u,v,w,z\}$, $E:=\{uv,us,uz,vz,vs,vw,ws,wz,zs\}$, $A:=\{uv,us,zu,vz,sv,vw,ws,zw,sz\}$. It is easy to check that $\vec{G}\in\mathcal{O}_b(G)$. In particular $\lambda_{\vec{G}}(v,z)=\lambda_{\vec{G}}(z,v)=2$. Suppose that for some $(r,t)\in\{(u,z),(u,v),(w,z),(w,v)\}$, (12) is satisfied. Then, by (12), $3=\varrho_{\vec{G}_{rt}}(\{r,t\})+\delta_{\vec{G}_{rt}}(\{r,t\})\geq\lambda_{\vec{G}_{rt}}(v,z)+\lambda_{\vec{G}_{rt}}(z,v)\geq\lambda_{\vec{G}}(v,z)+\lambda_{\vec{G}}(z,v)=4$, a contradiction. □

We note that the example given above also provides a counter-example to a conjecture of Jackson containing fewer vertices than previous examples due to Enni, for details see [6].

Question 3 (Open Problem)**.** *If $\vec{G}$ is a best-balanced orientation of $G=(V+s,E)$ and $\varrho_{\vec{G}}(s)=\delta_{\vec{G}}(s)$ then there exist $rs,st\in A(\vec{G})$ so that $\vec{G}_{rt}$ is a best-balanced orientation of G_{rt}.*

Though Question 3 is an open problem, a related question can be answered affirmatively.

Theorem 6.1. *For every pair rs,st of edges of a graph $G=(V+s,E)$ there exists a best-balanced orientation $\vec{G}$ of G so that $rs,st\in A(\vec{G})$ and $\vec{G}_{rt}$ is a best-balanced orientation of G_{rt}.*

Proof. By Theorem 3.1, there exists a feasible pairing M of G. Then M is a pairing of G_{rt} and hence, by Theorem 3.7, $G_{rt}+M$ has an Eulerian orientation $\vec{G}_{rt}+\vec{M}$ so that $\vec{G}_{rt}\in\mathcal{O}_b(G_{rt})$ (we can assume that rt is directed as $\vec{rt}$ in $\vec{G}_{rt}$). Then, for $\vec{G}:=\vec{G}_{rt}-rt+rs+st$, $\vec{G}+\vec{M}$ is Eulerian, that is, since $M\in P_f(G)$, $\vec{G}\in\mathcal{O}_b(G)$. □

For another similar problem we have negative answer.

Question 4. *For every graph $G=(V+s,E)$ with $d(s)\geq 4$ there exist $rs,st\in E$ such that for every best-balanced orientation $\vec{G}_{rt}$ of G_{rt} with $rt\in A(\vec{G}_{rt})$, $\vec{G}:=\vec{G}_{rt}-rt+rs+st$ is a best-balanced orientation of G.*

Counter-example 4. In Figure 4 (a) the graph $G=(V+s,E)$ is given. By symmetry there are two different choices of the pair $\{r,t\}$. Figure 4 (b) and (c) show best-balanced orientations $\vec{G}_{rt}$ for the two corresponding choices. A cut indicated in Figure 4 (b) and (c) has the property $\delta(X)=1$ in both cases, consequently $\vec{G}:=\vec{G}_{rt}-rt+rs+st$ cannot be best-balanced because $\lambda_{\vec{G}}(s,z)=1<2=\lambda_G(s,z)/2$.

7 Simultaneous well-balanced orientations

In this section we consider some possible generalizations of Theorem 3.3 and Theorem 3.8. Here we consider the statements of these theorems as assuring simultaneous (compatible) best-balanced orientations of some graphs.

The first two questions correspond to the local and global cases related to Theorem 3.3, i.e. the subgraph-chain property.

Question 5. *Let G_3 be a subgraph of G_2 and G_2 a subgraph of G_1. There exist orientations $\vec{G}_i$ of G_i for $i=1,2,3$, such that $\vec{G}_j$ is a restriction of $\vec{G}_i$ if $j>i$, and for all i*

(a) Local case: $\vec{G}_i\in\mathcal{O}_w(G_i)$.

(b) Global case: *if G_i is $2k_i$-edge-connected for some integers k_1,k_2,k_3 then $\vec{G}_i$ is a k_i-arc-connected orientation of G_i.*

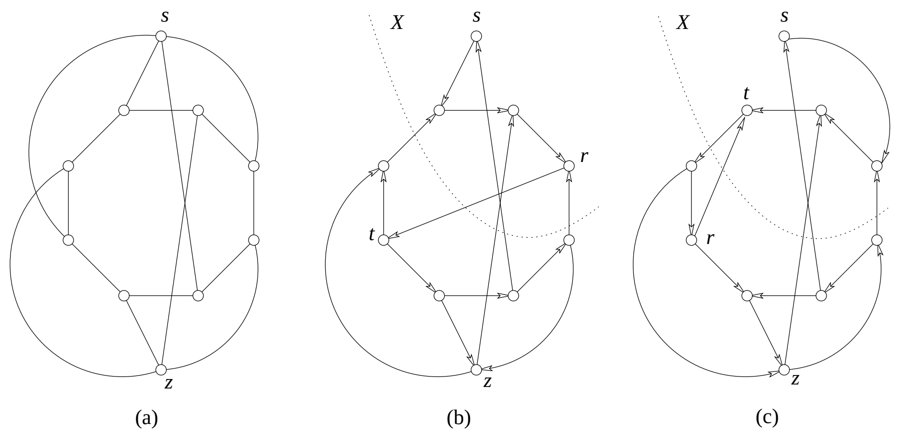

Figure 4: Counterexample for Question 4

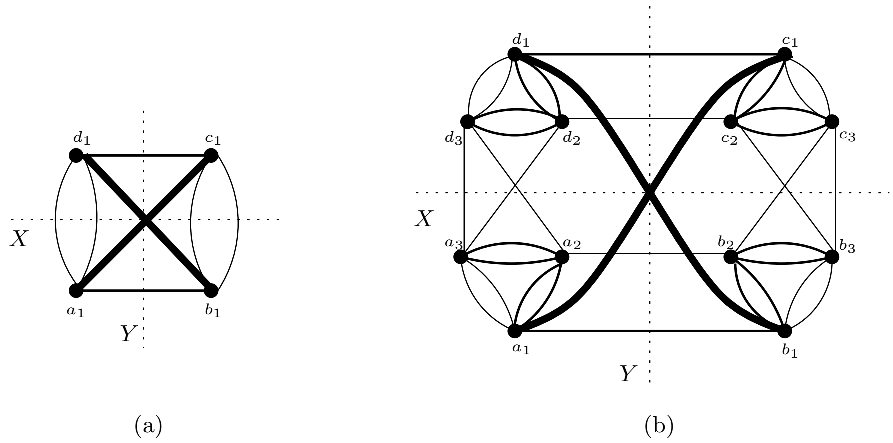

Figure 5: Counterexamples to the subgraph-chain property

Counter-examples 5. Let $G_i := (V_i, E_i)$ $(i = 1, 2, 3)$ be defined as in Figure 5, that is

(a) $V_1 = V_2 = V_3 := \{a_1, b_1, c_1, d_1\}$, $E_3 := \{a_1d_1, a_1d_1, b_1c_1, b_1c_1\}$, $E_2 := E_3 \cup \{a_1b_1, c_1d_1\}$, $E_1 := E_2 \cup \cup \{a_1c_1, b_1d_1\}$. Let $X := \{a_1, b_1\}, Y := \{a_1, d_1\}$.

(b) $V_1 = V_2 = V_3 := \{a_1, a_2, a_3, b_1, b_2, b_3, c_1, c_2, c_3, d_1, d_2, d_3\}$, $E_3 := \{a_2b_2, b_2c_3, c_3b_3, b_3c_2, c_2d_2, d_2a_3, a_3d_3, d_3a_2\} \cup \{x_3x_1, x_3x_1 : x \in \{a, b, c, d\}\}$, $E_2 := E_3 \cup \{a_1b_1, c_1d_1\} \cup \{x_1x_2, x_1x_2, x_2x_3, x_2x_3 : x \in \{a, b, c, d\}\}$, $E_1 := E_2 \cup \{a_1c_1, b_1d_1\}$ and $k_3 = 1$, $k_2 = 2$, $k_1 = 3$. Let $X := \{a_1, a_2, a_3, b_1, b_2, b_3\}$, $Y := \{a_1, a_2, a_3, d_1, d_2, d_3\}$.

We prove at the same time for (a) and (b) that the required orientations do not exist. Suppose that they do exist. It is easy to check that $\vec{G}_1$ and $\vec{G}_3$ are Eulerian orientations of G_1 and G_3, whence, by (1), $f_{\vec{G}_1}(X) = 0 = f_{\vec{G}_1}(Y)$ and $f_{\vec{G}_3}(X) = 0$. G_2 is $2k$-edge-connected and $d_{G_2}(Y) = 2k$, so $f_{\vec{G}_2}(Y) = 0$. Then $f_{\vec{G}_1 - \vec{G}_2}(X) = f_{\vec{G}_1 - \vec{G}_3}(X) = f_{\vec{G}_1}(X) - f_{\vec{G}_3}(X) = 0$ and $f_{\vec{G}_1 - \vec{G}_2}(Y) = f_{\vec{G}_1}(Y) - f_{\vec{G}_2}(Y) = 0$. Note that $E(G_1 - G_2) = E_1 - E_2 = \{a_1c_1, b_1d_1\}$ and $a_1 \in X \cap Y$, $c_1 \in V - (X \cup Y)$, $b_1 \in X - Y$, $d_1 \in Y - X$, a contradiction. $\square$

Regarding general simultaneous orientations, we may ask the following question:

Question 6. *Given two graphs (neither edge-disjoint nor containing each other), is there a good characterization for having simultaneous best-balanced orientations?*

The next theorem and corollary show that this problem is NP-complete even for Eulerian graphs.

Theorem 7.1. *Deciding whether two Eulerian graphs, $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ have Eulerian orientations that agree on the common edges $E_1 \cap E_2$, is NP-complete.*

Proof. The problem is clearly in NP. For the completeness we show a reduction from ONE-IN-THREE 3SAT (see [14], Problem LO4). For a given 3SAT formula, n denotes the number of variables, the clauses are denoted by $\mathcal{C}_1, \mathcal{C}_2 \dots, \mathcal{C}_m$, and J_i denotes the set of indices of the clauses that contain the variable x_i (we assume that every clause contains 3 different variables).

Construct first the graph $G_1 = (V_1, E_1)$ as follows. Each connected component $G_1^i = (V_1^i, E_1^i)$ of G_1 corresponds to a clause $\mathcal{C}_i$, namely V_1^i contains the vertices $\{C_i, C_i'\}$ and the 6 vertices $\{x_j^i, \overline{x}_j^i : x_j$ or $\overline{x}_j$ occurs in $\mathcal{C}_i\}$ and E_1^i contains the edge C_iC_i', the edges $\{x_j^i\overline{x}_j^i : x_j^i \in V_1^i\}$, the edges $\{C_ix_j^i, C_i'\overline{x}_j^i\}$ if x_j occurs in $\mathcal{C}_i$ and the edges $\{C_i\overline{x}_j^i, C_i'x_j^i\}$ if $\overline{x}_j$ occurs in $\mathcal{C}_i$. Note that vertices corresponding to literals are of degree-two and vertices corresponding to clauses are of degree four.

The graph $G_2 = (V_2, E_2)$ is constructed in such a way that each connected component of G_2 is a cycle of even length. One cycle contains $C_1, C_1', C_2, C_2', \dots C_m, C_m'$ in this order. We also have cycles for every variable: for any $1 \le i \le n$ there is a cycle on $x_i^{j_1}, \overline{x}_i^{j_1}, x_i^{j_2}, \overline{x}_i^{j_2}, \dots, x_i^{j_i}, \overline{x}_i^{j_i}$ in this order, where $\{j_1, j_2, \dots, j_i\}$ is the set of the indices of the clauses that contain variable x_i (or its negated).

First we claim that if there is a truth assignment such that in each clause exactly one literal is TRUE then the required Eulerian orientations exist. Orient first G_2, it is enough to declare the orientation of one edge in each cycle. Let $C_1'C_1$ be oriented from C_1' to C_1, and for each i let the edge $x_i^j\overline{x}_i^j$ (for any $j \in J_i$) be oriented from the FALSE value to the TRUE value. Now G_1 has a unique orientation that extends the orientation of the common edges and that makes each vertex of degree two Eulerian. Since each clause $\mathcal{C}_i$ contains exactly one literal of value TRUE, this orientation is Eulerian.

On the other hand suppose that we have Eulerian orientations $\vec{G}_1$ and $\vec{G}_2$ that agree on the common edges. If edge C_1C_1' is oriented from C_1 to C_1' then reverse both Eulerian orientations. The Eulerian orientation $\vec{G}_2$ first ensures that $C_i'C_i$ is a directed edge for all i. Second, it also ensures that for all i either $x_i^j\overline{x}_i^j$ is a directed edge for all j, or $\overline{x}_i^jx_i^j$ is a directed edge for all j. Assign the value TRUE to variable x_i iff $\overline{x}_i^jx_i^j$ is a directed edge. We claim that this assignment makes TRUE exactly one literal in each clause. Indeed, from the three edges between C_i and the three literal-copies exactly one is directed towards C_i, and exactly the corresponding literal has value TRUE. $\square$

We remark that another construction can be made by adding some extra vertices, in which both graphs are connected: add vertices y_i and z_i to graph G_i (for $i = 1, 2$), and connect them to one degree two vertex of each component, then connect y_i to z_i if they have odd degree. To see that the above reasoning goes through, observe that if both y_iv and z_iv is directed towards v then there is no Eulerian orientation extending it.

Corollary 7.2. *Deciding whether two graphs have simultaneous best-balanced orientations is NP-complete.*

8 Feasible pairing defining a best-balanced orientation

Nash-Williams' original idea was that every feasible pairing provides a best-balanced orientation. In fact every feasible pairing provides lots of best-balanced orientations. A natural question is whether every best-balanced orientation can be defined by a feasible pairing.

Question 7. *For every best-balanced orientation $\vec{G}$ of G there exists a feasible pairing M and an orientation $\vec{M}$ of M such that $\vec{G} + \vec{M}$ is Eulerian.*

Counter-example 7. Let $G = (V, E)$ and $\vec{G} = (V, A)$ be defined as follows (see Figure 6): $V := \{a, b, c, p, q, r, x, y, z\}$, $E := \{ap, aq, ar, bp, bq, br, cx, cy, cz, xp, py, yq, qz, zr, rx\}$, $A := \{ap, qa, ra, bp, qb, rb, xc, yc, cz, px, py, yq, zq, zr, xr\}$. It is easy to check that $\vec{G} \in \mathcal{O}_b(G)$.

We show that if $M \in \mathcal{P}_f(G)$, then $ab \in M$. Note that $T_G = \{a, b, c, x, y, z\}$. Let $X := \{a, b, p, r, x\}$, $Y := \{a, b, p, q, y\}$, $Z := \{a, b, q, r, z\}$. Note that $d_G(W) = 5$ and $\hat{R}(W) = 4$ hence $b_G(W) = 1$ for $W \in \{X, Y, Z\}$. Then, by (5) and (7), $3 = b_G(X) + b_G(Y) + b_G(Z) \ge d_M(X) + d_M(Y) + d_M(Z) \ge d_M(X \cap Y \cap Z) + d_M(X - (Y \cup Z)) + d_M(Y - (X \cup Z)) + d_M(Z - (X \cup Y)) = d_M(\{a, b\}) + d_M(x) + d_M(y) + d_M(z) \ge 0 + 1 + 1 + 1 = 3$, so $d_M(\{a, b\}) = 0$ that is $ab \in M$.

Then for every orientation $\vec{M}$ of any feasible pairing M of G either $\delta_{\vec{M}}(a) = 0$ or $\delta_{\vec{M}}(b) = 0$. Then, since $f_{\vec{G}}(a) = f_{\vec{G}}(b) = 1$, $\vec{G} + \vec{M}$ cannot be Eulerian. $\square$

The following Theorem 8.2 shows that the answer for Question 7 is affirmative for global edge-connectivity. For the proof we need the following stronger version of Mader's splitting-off theorem [21] due to Frank [11].

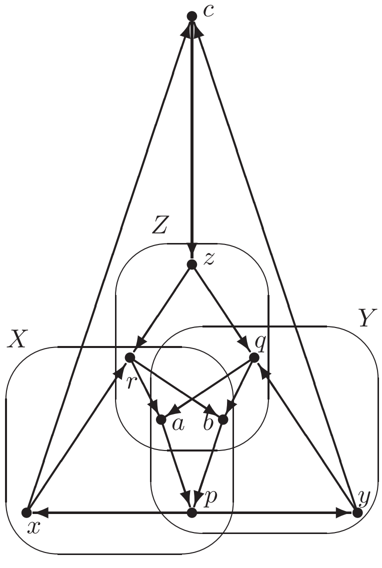

Figure 6: A best-balanced orientation which can not be defined with a feasible pairing

Theorem 8.1. *Let $\vec{H} = (U+s, F)$ be k-arc-connected in U. If $\delta_{\vec{H}}(s) - \varrho_{\vec{H}}(s) < \varrho_{\vec{H}}(s) < 2\delta_{\vec{H}}(s)$ then there exist $rs, st \in F$ so that $\vec{H}_{rt}$ is k-arc-connected in U.*

Theorem 8.2. *Let $G = (V, E)$ be a $2k$-edge-connected graph and let $\vec{G} = (V, A)$ be a smooth k-arc-connected orientation of G. Then there is a pairing M of G and an orientation $\vec{M}$ of M so that*

$$d_M(X) \leq d_G(X) - 2k \quad \forall\, \emptyset \neq X \subsetneq V \text{ and} \tag{13}$$

$$\vec{G} + \vec{M} \text{ is Eulerian.} \tag{14}$$

Proof. By induction on $|A|$.
Case 1 If there is $s \in V$ with $d(s) \geq 2k+2$. Then, by (3) and Theorem 8.1, there exist $rs, st \in A$ so that $\vec{G}_{rt}$ is k-arc-connected in $V - s$. It follows, by the assumption of Case 1 and (3), that $\vec{G}_{rt}$ is k-arc-connected. Note that $T_{G_{rt}} = T_G$. $|A(\vec{G}_{rt})| < |A|$ so by induction there is a pairing M of G_{rt} and an orientation $\vec{M}$ of M so that (13) and (14) are satisfied for (G_{rt}, M) and for $(\vec{G}_{rt}, \vec{M})$ and hence for (G, M) and for $(\vec{G}, \vec{M})$ and we are done.

Case 2 If there is $s \in V$ with $d(s) = 2k$. This case can be handled in the same way as Case 1 but here we have to make a complete splitting off at s.

Case 3 Otherwise, $d(s) = 2k+1$ for all $s \in V$. Then $T_G = V$. By a result of Mader [19], since there is no vertex v with $\varrho_{\vec{G}}(v) = \delta_{\vec{G}}(v)$, there exists $uv \in A$ such that $\vec{G}' := \vec{G} - uv$ is k-arc-connected, obviously u and v have in-degree and out-degree k in $\vec{G}'$, so $\vec{G}'$ is also smooth. As $|A(\vec{G}')| < |A|$, by induction there is a pairing M' on $T_{G'} = T_G - \{u, v\}$ and an orientation $\vec{M}'$ of M' so that (13) and (14) are satisfied for (G', M') and for $(\vec{G}', \vec{M}')$. Let $M := M' + uv$ and $\vec{M} := \vec{M}' + vu$. Then $\vec{G} + \vec{M} = (\vec{G}' + \vec{M}') + uv + vu$ is Eulerian. Moreover, for all $\emptyset \neq X \subsetneq V$ either $d_M(X) = d_{M'}(X)$ and $d_G(X) = d_{G'}(X)$ or $d_M(X) = d_{M'}(X) + 1$ and $d_G(X) = d_{G'}(X) + 1$ so (13) is satisfied for G and M. $\square$

9 The structure of feasible pairings

We call a feasible pairing M a **feasible matching** (in G), if for every edge uv of M, uv is also an edge of G. A k-feasible matching is defined analogously for a positive integer k.

Theorem 9.1. *Deciding whether a graph has a feasible matching is NP-complete, even for planar three-regular graphs.*

Proof. We claim that a 2-connected 3-regular graph $G = (V, E)$ has a feasible matching if and only if G is Hamiltonian. Indeed, for a perfect matching M of G, the 2-regular graph $G - M$ is Hamiltonian if and only if $G - M$ is 2-edge-connected that is if and only if $d_{G-M}(X) \geq 2$ for all $\emptyset \neq X \subsetneq V$ or equivalently if M is feasible.

It is known that deciding whether a graph has a Hamiltonian cycle is NP-complete even for planar 2-connected 3-regular graphs [15]. □

Corollary 9.2. *We are given a graph G and a weight on each pair of distinct odd-degree vertices. Finding the minimum weight strong pairing is NP-hard, even for planar 3-regular graphs and for 0–1-valued weighting.*

We mention that the proof given here shows that the minimum weight feasible pairing problem and the feasible matching problem are NP-hard *even for the global case with* $k = 1$ (i.e. it is NP-hard to find a minimum weight 1-feasible pairing or to decide whether there is a 1-feasible matching in a given graph).

10 Feasible pairing for connectivity functions

A set function $b : 2^V \to \mathbb{R}$ is called **skew-submodular** if for every $X, Y \subseteq V$, at least one of the following two inequalities is satisfied:

$$b(X) + b(Y) \geq b(X \cap Y) + b(X \cup Y), \tag{15}$$

$$b(X) + b(Y) \geq b(X - Y) + b(Y - X). \tag{16}$$

A set function p is called **skew-supermodular** if $-p$ is skew-submodular. We mention that, by [22], $\hat{R}_G$ is skew-supermodular and hence b_G is skew-submodular. A set function b on V is called **crossing submodular** if (15) is satisfied for every $X, Y \subseteq V$ with $X \cap Y,\ X - Y,\ Y - X,\ V - (X \cup Y) \neq \emptyset$. For a set function b we define $T_b = \{v : b(v) \text{ is odd}\}$.

Question 8. *Let $b : 2^V \to \mathbb{Z}_0^+$ be a symmetric, skew-submodular function with $b(\emptyset) = 0$ and $b(X) \equiv |X \cap T_b|$ mod 2. Then there exists a pairing M on T_b that satisfies*

$$d_M(X) \leq b(X) \qquad \forall\ X \subseteq V. \tag{17}$$

Counter-example 8. Let $b(X)$ be defined on V with $|V| = 6$ as follows: $b(X) := 0$ if $X = \emptyset$, or $X = V$; 1 if $|X|$ is odd and 2 otherwise. It is easy to see that b satisfies all the conditions. Note that $T_b = V$. For any pairing M on T_b, we may choose $X \subsetneq V$ with $d_M(X) = 3$ but then X violates (17). □

Note that, by Theorem 3.1, the answer for Question 8 is affirmative for $b(X) = b_G(X)$.

The problem corresponding to the global case is the following open problem.

Question 9 (Open Problem)**.** *Let $b : 2^V \to \mathbb{Z}_0^+$ be a symmetric crossing submodular function with $b(\emptyset) = 0$ and $b(X) \equiv |X \cap T_b|$ mod 2. Then there exists a pairing M on T_b that satisfies (17).*

If the answer to Question 9 was affirmative then it would imply the following theorem that can be proved directly.

Theorem 10.1. *Let $G = (V, E)$ be an undirected graph. Let $b : 2^V \to \mathbb{Z}_0^+$ be a crossing submodular function with $b(X) + d_G(X)$ even for every $X \subseteq V$. Then there exists an orientation $\vec{G}$ of G satisfying*

$$f_{\vec{G}}(X) \quad \leq \quad b(X) \quad \forall\ X \subseteq V. \tag{18}$$

Proof. Let $\vec{G} = (V, A)$ be an arbitrary orientation of G. Let $P := \{z \in \mathbb{R}^A : 0 \leq z(a) \leq 1\ \forall a \in A,\ \delta^z_{\vec{G}}(X) - \varrho^z_{\vec{G}}(X) \leq (b(X) - f_{\vec{G}}(X))/2\ \ \forall X \subseteq V\}$. By the modularity of $f_{\vec{G}}$ and by the assumptions, $(b(X) - f_{\vec{G}}(X))/2$ is integral and crossing submodular. Then, by the Edmonds-Giles theorem [4], P is an integral polyhedron. The vector $\frac{1}{2}\mathbb{1}$ belongs to P because b is non-negative. Then P contains an integral vector $\overline{z}$. Let $\vec{G}'$ be the orientation obtained from $\vec{G}$ by reversing the arcs $a \in A(\vec{G})$ for which $\overline{z}(a) = 1$. Then, since $\overline{z}$ is a $0-1$ vector in P, $f_{\vec{G}'}(X) = \varrho_{\vec{G}'}(X) - \delta_{\vec{G}'}(X) = (\varrho_{\vec{G}}(X) - \varrho^{\overline{z}}_{\vec{G}}(X) + \delta^{\overline{z}}_{\vec{G}}(X)) - (\delta_{\vec{G}}(X) - \delta^{\overline{z}}_{\vec{G}}(X) + \varrho^{\overline{z}}_{\vec{G}}(X)) = f_{\vec{G}}(X) + 2(\delta^{\overline{z}}_{\vec{G}}(X) - \varrho^{\overline{z}}_{\vec{G}}(X)) \leq b(X)\ \ \forall X \subseteq V$, and hence $\vec{G}'$ is the desired orientation. □

Note that if G is $2k$-edge-connected and $b(X) = d_G(X) - 2k$ for all $\emptyset \neq X \subsetneq V$ and $b(\emptyset) = b(V) = 0$, then Theorem 10.1 is equivalent to Theorem 3.4. We remark that Theorem 10.1 also follows from a theorem of Frank [7] on orientations covering a G-supermodular function.

Question 10. *Let $d : 2^V \to \mathbb{Z}_0^+$ be a symmetric function that satisfies $d(\emptyset) = 0$ and for all $X, Y \subseteq V$*

$$d(X) + d(Y) + d(X \triangle Y) = d(X \cap Y) + d(X \cup Y) + d(X - Y) + d(Y - X), \tag{19}$$

$$d(X) + d(Y) - d(X \cup Y) \text{ is even if } X \cap Y = \emptyset. \tag{20}$$

Let $\hat{R} : 2^V \to \mathbb{Z}_0^+$ be an even valued, symmetric, skew-supermodular function. Suppose that $\hat{R}(X) \leq d(X)$ for all $X \subseteq V$. Then there exists a pairing M on T_d that satisfies

$$d_M(X) \leq d(X) - \hat{R}(X) \qquad \forall\ X \subseteq V. \tag{21}$$

Counter-example 10. Let $V := \{u, v, w, z\}$, $G := (V, \{uw, uz, vw, vz, wz\})$, $H := (V, \{uv\})$, $d(X) := d_G(X) - d_H(X)$, $\hat{R}(X) := 2$ if $|X \cap \{w, z\}| = 1$, and 0 otherwise. Since for a proper subset X, $d_G(X) \geq 1$ and $d_H(X) \leq 1$, $d(X) \geq 0\ \ \forall X \subseteq V$. Clearly, d is integer valued and symmetric. d_G and d_H satisfy (19) and (20), consequently d also satisfies them. It is easy to see that $\hat{R}$ satisfies all the conditions. Note that $T_d = V$. Let M be an arbitrary pairing on T_d. Let e be the edge of M incident to w. Let $X := \{u, w\}$ and let $Y := \{v, w\}$. Then e leaves either X or Y but $d(X) - \hat{R}(X) = 0 = d(Y) - \hat{R}(Y)$ so either X or Y violates (21). □

Note that, by Theorem 3.1, the answer for Question 10 is affirmative for $d(X) = d_G(X)$ and $\hat{R}(X) = \hat{R}_G(X)$.

Question 11 (Open Problem)**.** *Let $G = (V, E)$ be a graph and $\hat{R} : 2^V \to \mathbb{Z}_0^+$ an even valued, symmetric, skew-supermodular function. Suppose that $\hat{R}(X) \leq d_G(X)\ \ \forall X \subseteq V$. Then there exists a pairing M on T_G that satisfies*

$$d_M(X) \leq d_G(X) - \hat{R}(X) \qquad \forall\ X \subseteq V. \tag{22}$$

Question 11 is an open problem. If $\hat{R}$ satisfies $\hat{R}(X \cup Y) \leq \max\{\hat{R}(X), \hat{R}(Y)\}$ for all $X, Y \subseteq V$ then $\hat{R}(X) = \max\{\mathbf{r}(x, y) :\ x \in X,\ y \in V - X\}$ for some symmetric, even valued $\mathbf{r} : V \times V \to \mathbb{Z}_0^+$ and hence, by Theorem 3.1, such a pairing exists.

11 Matroid property

If $\vec{G}$ is an orientation of G then let $T^+_{\vec{G}} := \{v \in V(G) : \varrho_{\vec{G}}(v) > \delta_{\vec{G}}(v)\}$. Note that if $\vec{G}$ is smooth, then $|T^+_{\vec{G}}| = |T_G|/2$.

The following *strict reorientation property* was proved for k-arc-connected orientations by Frank in [8]: if $\vec{G}_1$ and $\vec{G}_2$ are k-arc-connected orientations of a graph $G = (V, E)$ and $\varrho_{\vec{G}_1}(u) < \varrho_{\vec{G}_2}(u)$ at a vertex $u \in V$ then there exists a directed path in $\vec{G}_1$ from u to some vertex $v \in V$ with $\varrho_{\vec{G}_1}(v) > \varrho_{\vec{G}_2}(v)$ such that reversing this path in $\vec{G}_1$ results in a k-arc-connected digraph. This result has interesting consequences, for example when restricted to smooth k-arc-connected orientations (which is not destroyed by such a reorientation) then it is equivalent with the following statement: for a $2k$-edge-connected graph G the family $\mathcal{T} := \{T^+_{\vec{G}} : \vec{G}$ is a smooth k-arc-connected orientation of $G\}$ is the base family of a matroid. Another consequence of the strict reorientation property is that k-arc-connected orientations of a graph satisfy the so called *linkage property.* In this section we investigate whether any of the above properties hold for well-balanced orientations.

First we investigate the matroid property:

Question 12. *$\mathcal{T} := \{T^+_{\vec{G}} : \vec{G} \in \mathcal{O}_b(G)\}$ is the base family of a matroid.*

Counter-example 12. Let G, $\vec{G}$, X, Y and Z be as in Figure 6. Then $\overleftarrow{G} \in \mathcal{O}_b(G)$ hence $B_1 := \{a, b, c\}$ and $B_2 := \{x, y, z\}$ are in $\mathcal{T}$. Suppose that $\mathcal{T}$ is the base family of a matroid. Then for $c \in B_1 - B_2$ there must exist $u \in B_2 - B_1$ such that $B_1 - \{c\} + \{u\} \in \mathcal{T}$, by symmetry we may suppose that $\{a, b, x\} \in \mathcal{T}$. Then there exists $\vec{G}' \in \mathcal{O}_b(G)$ so that $T^+_{\vec{G}'} = \{a, b, x\}$. Whence, by (10) and (1), $1 = b_G(X) \geq f_{\vec{G}'}(X) = \sum_{u \in X} f_{\vec{G}'}(u) = 3$, a contradiction. □

We can see from the previous proof that the reorientation property in the strict sense as introduced above is not true for well-balanced orientations. Furthermore, a weaker reorientation property was shown not to hold in [1], namely the following statement was disproved there: if $\vec{G}_1, \vec{G}_2 \in \mathcal{O}_w(G)$ such that there is an $x \in V(G)$ with $\varrho_{\vec{G}_1}(x) \neq \varrho_{\vec{G}_2}(x)$ then there exist $u, v \in V(G)$ with $\varrho_{\vec{G}_1}(u) < \varrho_{\vec{G}_2}(u)$ and $\varrho_{\vec{G}_1}(v) > \varrho_{\vec{G}_2}(v)$ such that reversing a directed path in $\vec{G}_1$ from u to v results in another well-balanced orientation. We formulate here an even weaker reorientation property and pose the following question:

Question 13 (Open Problem)**.** *Let $\vec{G}^a, \vec{G}^b \in \mathcal{O}_w(G)$. Then there exist $\vec{G}^0 = \vec{G}^a, \vec{G}^1, ..., \vec{G}^l = \vec{G}^b$ such that $\vec{G}^k \in \mathcal{O}_w(G)$ and $\vec{G}^k$ is obtained from $\vec{G}^{k-1}$ by reversing a directed path or a directed cycle $(1 \le k \le l)$.*

This is an open problem, but it is known that, by Frank [8], the answer for Question 13 is affirmative for the case of global edge-connectivity.

Now we investigate whether the linkage property holds for well-balanced orientations.

Question 14. *Let $l, u : V \to \mathbb{Z}_0^+$ such that $l(v) \le u(v)$ for all $v \in V$. Then there exists $\vec{G} \in \mathcal{O}_w(G)$ such that $l(v) \le \varrho_{\vec{G}}(v) \le u(v)$ $\forall v \in V$ if and only if there exist $\vec{G}^1, \vec{G}^2 \in \mathcal{O}_w(G)$ such that $l(v) \le \varrho_{\vec{G}^1}(v)$ $\forall v \in V$ and $\varrho_{\vec{G}^2}(v) \le u(v)$ $\forall v \in V$.*

Counter-example 14. Let G, $\vec{G}^1 := \vec{G}$, $\vec{G}^2 := \overleftarrow{G} \in \mathcal{O}_w(G)$, X, Y and Z as in Figure 6. Let the functions l and u be defined as follows: $l(a) = l(b) =: 2$ and $l(t) := \lfloor \frac{d_G(t)}{2} \rfloor$ $\forall t \in V - a - b$, $u(c) := 1$ and $u(t) := \lceil \frac{d_G(t)}{2} \rceil$ $\forall t \in V - c$. Then $l(v) \le \varrho_{\vec{G}^1}(v)$ $\forall v \in V$ and $\varrho_{\vec{G}^2}(v) \le u(v)$ $\forall v \in V$. Let $\vec{G}^3 \in \mathcal{O}_w(G)$ such that $l(v) \le \varrho_{\vec{G}^3}(v)$ $\forall v \in V$. Recall that $b_G(X) = b_G(Y) = b_G(Z) = 1$. Then, by Claim 3.5, $1 = b_G(X) \ge f_{\vec{G}^3}(X) = f_{\vec{G}^3}(x) + f_{\vec{G}^3}(p) + f_{\vec{G}^3}(a) + f_{\vec{G}^3}(b) + f_{\vec{G}^3}(r) = f_{\vec{G}^3}(x) + 0 + 1 + 1 + 0$, so $f_{\vec{G}^3}(x) \le -1$ and hence $f_{\vec{G}^3}(x) = -1$. Similarly, $f_{\vec{G}^3}(y) = f_{\vec{G}^3}(z) = -1$. Then, since $f_{\vec{G}^3}(V) = 0$, $f_{\vec{G}^3}(c) = 1$, that is $\varrho_{\vec{G}^3}(c) = 2 > 1 = u(c)$. Thus there is no well-balanced orientation of G whose in-degree function satisfies both the lower and upper bounds. □

Question 14 is valid for the global case by Frank [7]. This follows from the facts that the in-degree vectors of k-arc-connected orientations form a base-polyhedron and for such polyhedra the linkage property holds. As mentioned above, this also follows easily from the strict reorientation property.

References

[1] J. Bang-Jensen, T. Jordán, and Z. Szigeti, unpublished results.

[2] A. Bernáth, *Hardness results for well-balanced orientations*, Tech. Report TR-2006-05, Egerváry Research Group, Budapest, 2006, `www.cs.elte.hu/egres`.

[3] A. Bernáth and G. Joret, *Well-balanced orientations of mixed graphs*, Tech. Report QP-2007-01, Egerváry Research Group, Budapest, 2007, `www.cs.elte.hu/egres` (To appear at Inform. Process. Lett.).

[4] J. Edmonds and R. Giles, *A min-max relation for submodular functions on graphs*, Studies in integer programming (Proc. Workshop, Bonn, 1975), North-Holland, Amsterdam, 1977, pp. 185–204. Ann. of Discrete Math., Vol. 1.

[5] Egres Open Problems, Problem 29
`http://www.cs.elte.hu/egres/problems/prob_29/prob_29.html`.

[6] S. Enni, *A note on mixed graphs and directed splitting off*, J. Graph Theory **27** (1998), no. 4, 213–221.

[7] A. Frank, *On the orientation of graphs*, J. Combin. Theory Ser. B **28** (1980), no. 3, 251–261.

[8] ______, *A note on k-strongly connected orientations of an undirected graph*, Discrete Math. **39** (1982), no. 1, 103–104.

[9] ______, *On a theorem of Mader*, Discrete Math. **101** (1992), no. 1-3, 49–57, Special volume to mark the centennial of Julius Petersen's "Die Theorie der regulären Graphs", Part II.

[10] ______, *Applications of submodular functions*, Surveys in combinatorics, 1993 (Keele), London Math. Soc. Lecture Note Ser., vol. 187, Cambridge Univ. Press, Cambridge, 1993, pp. 85–136.

[11] ______, *Connectivity augmentation problems in network design*, Mathematical Programming: State of the Art 1994 (1994), 34–63.

[12] ______, *Orientations of graphs and submodular flows*, Congr. Numer. **113** (1996), 111–142, Festschrift for C. St. J. A. Nash-Williams.

[13] H. N. Gabow, *Efficient splitting off algorithms for graphs*, STOC '94: Proceedings of the twenty-sixth annual ACM symposium on Theory of computing (New York, NY, USA), ACM Press, 1994, pp. 696–705.

[14] M. R. Garey and D. S. Johnson, *Computers and intractability: A guide to the theory of NP-completeness*, W. H. Freeman & Co., New York, NY, USA, 1979.

[15] M. R. Garey, D. S. Johnson, and R. E. Tarjan, *The planar Hamiltonian circuit problem is NP-complete*, SIAM J. Comput. **5(4)** (1976), 704–714.

[16] S. Iwata, T. Király, Z. Király, and Z. Szigeti, *On well-balanced orientations, counter-examples for related problems*, Tech. Report TR-2004-16, Egerváry Research Group, Budapest, 2004, `www.cs.elte.hu/egres`.

[17] Z. Király and Z. Szigeti, *Simultaneous well-balanced orientations of graphs*, J. Combin. Theory Ser. B **96** (2006), no. 5, 684–692.

[18] L. Lovász, *Combinatorial problems and exercises*, North-Holland Publishing Co., Amsterdam, 1979.

[19] W. Mader, *Ecken vom Innen- und Aussengrad n in minimal n-fach kantenzusammenhängenden Digraphen*, Arch. Math. (Basel) **25** (1974), 107–112.

[20] ———, *A reduction method for edge-connectivity in graphs*, Ann. Discrete Math. **3** (1978), 145–164, Advances in graph theory (Cambridge Combinatorial Conf., Trinity College, Cambridge, 1977).

[21] ———, *Konstruktion aller n-fach kantenzusammenhängenden Digraphen*, European J. Combin. **3** (1982), no. 1, 63–67.

[22] C. St. J. A. Nash-Williams, *On orientations, connectivity and odd vertex pairings in finite graphs*, Canadian J. Math. **12** (1960), 555–567.

[23] ———, *Well-balanced orientations of finite graphs and unobtrusive odd-vertex-pairings*, Recent Progress of Combinatorics (1969), 133–149.

[24] É. Tardos, personal communication.

Les cahiers Leibniz ont pour vocation la diffusion des rapports de recherche, des séminaires ou des projets de publication sur des problèmes liés au mathématiques discrètes.

Pour soummettre un articles dans les cahiers,
`http://www.g-scop.inpg.fr/CahiersLeibniz/`

Responsables de la publication : Nadia Brauner et András Sebő
ISSN : 1298-020X