Les cahiers Leibniz

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On b-colorings in regular graphs*

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Abstract

A b-coloring is a coloring of the vertices of a graph such that each color class contains a vertex that has a neighbor in all other color classes. El-Sahili and Kouider have conjectured that every d-regular graph with girth at least 5 has a b-coloring with $d+1$ colors. We show that the Petersen graph infirms this conjecture, and we propose a new formulation of this question and give a positive answers for small degree.
Keywords: Coloration, b-coloring, a-chromatic number, b-chromatic number.

1 Introduction

A proper coloring of a graph $G = (V, E)$ is a mapping c from V to the set of positive integers (colors) such that any two adjacent vertices are mapped to different colors. Each set of vertices colored with one color is a stable set of vertices of G, so a coloring is a partition of V into stable sets. The smallest number k for which G admits a coloring with k colors is the chromatic number $\chi(G)$ of G.

Many graph invariants related to colorings have been defined. Most of them try to minimize the number of colors used to color the vertices under some constraints. For some other invariants, it is meaningful to try to maximize this number. The b-chromatic number is such an example. When we try to color the vertices of a graph, a simple trick consists in starting form a coloring and trying to decrease the number of colors by reducing or merging color classes. This motivated the introduction of *a-colorings* and the *a-chromatic number* by Harary, Hedetniemi and Prins [5]. An a-coloring is a proper coloring such that the subgraph induced by any two different color classes contains an edge. Clearly, the process of reducing or merging suggested above is impossible when we start from an a-coloring. So the a-coloring number is a measure of how hard it is to obtain a coloring with few colors. This inspired Irving and Manlove [7, 8] to introduce another procedure, which consists, given a coloring, in trying to reduce the number of colors by transfering every vertex from

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a fixed color class to another color class. A *b-coloring* is a proper coloring such that every color class i contains at least one vertex that has a neighbor in all the other classes. Any such vertex will be called a *b-dominating vertex* of color i. The *b-chromatic number* $b(G)$ is the largest integer k such that G admits a b-coloring with k colors.

Clearly every $\chi(G)$-coloring of a graph G is a b-coloring, and so $\chi(G) \leq b(G)$. The behavior of the b-chromatic number can be surprising. For example, the values of k such that a graph admits a b-coloring with k colors do not necessarily form an interval of the set of integers; in fact any finite subset of $\{2, \ldots\}$ can be the set of these values for some graph [3].

Irving and Manlove [7, 8] proved that deciding whether a graph G admits a b-coloring with a given number of colors is an NP-complete problem, even when it is restricted to the class of bipartite graphs. On the other hand, they proved that the problem is polynomially solvable for trees. This NP-completeness results has incited researchers to establish bounds on the b-chromatic number in general or to find its exact values for subclasses of graphs.

For any vertex v of a graph G, the *neighborhood* of v is the set $N(v) = \{u \in V(G) \mid uv \in E\}$ and the *degree* of v is is $\deg(v) = |N(v)|$. A graph G is d-regular if every vertex of G has degree equal to d. For integer $k \geq 3$, we let C_k denote the cycle with k vertices. The *girth* of G is the length of a shortest cycle in G. In [2], El-Sahili and Kouider pose the following question: *Is it true that every d-regular graph G with girth $g(G) \geq 5$ satisfies $b(G) = d+1$?* We discovered that the Petersen graph offers a negative answer to this question. However, we also propose some positive results, in the case where $d \leq 6$, which suggest that the Petersen graph might be the only counterexample to the question.

Theorem 1 *The Petersen graph has b-chromatic number 3.*

Proof. Let G be the Petersen graph, with vertices $v_1, \ldots, v_5$, $w_1, \ldots, w_5$ such that v_1, v_2, v_3, v_4, v_5 induce a 5-cycle in this order, w_1, w_3, w_5, w_2, w_4 induce a 5-cycle in this order, and v_iw_i is an edge for each $i = 1, \ldots, 5$. First we show that G has a b-coloring with 3 colors. By assigning color 1 to v_1, v_4, w_2, w_3, color 2 to v_2, w_4, w_5, and color 3 to v_3, v_5, w_1, we obtain a coloring in which v_1, v_2, v_3 are b-dominating.

Now suppose that G admits a b-coloring with 4 colors. For $j = 1, \ldots, 4$, let d_j be a b-dominating vertex of color j, and let $D = \{d_1, \ldots, d_4\}$. Note that each vertex d_j must have exactly one neighbour of each of the three colors different from j.

Suppose that D induces a stable set. So, up to symmetry, we can assume that $D = \{v_1, v_3, w_4, w_5\}$, with $d_1 = v_1$, $d_3 = v_3$, $d_2 = w_4$, $d_4 = w_5$. Without loss of generality, v_2 has color 2, and so w_2 does not have color 2. Since v_1 is b-dominating, it must have a neighbor of color 4, which can only be w_1, and a neighbor of color 3, which can only be v_5. Since v_3 is b-dominating, it must have a neighbor of color 4, which can only be v_4, and a neighbor of color 1, which can only be w_3. But now w_5 cannot have a neighbor of color 2, a contradiction. So D does not induce a stable set.

We may assume that d_1, d_2 are adjacent, and, up to symmetry, that $d_1 = v_1$ and $d_2 = v_2$ (since all edges of the Petersen graph play the same role). Now, it is easy to see that, wherever d_3 may be, there is a C_5 of G that contains d_1, d_2, d_3. So (since all C_5's of the

Petersen graph play the same role), we can assume that d_3 is one of v_3, v_4, v_5. Up to symmetry, this leads to two cases.

Case 1: $d_3 = v_3$. Since v_2 is b-dominating, it has a neighbour of color 4, which can only be w_2. One of v_4, v_5, say v_5, does not have color 4. Since v_1 is b-dominating, it has a neighbour of color 4, which can only be w_1, and a neighbour of color 3, which can only be v_5. Since v_3 is b-dominating, it has a neighbour of color 4, which can only be v_4, and a neighbour of color 1, which can only be w_3. Note that w_5 can only have color 2. But now, the vertices of color 4 are w_1, w_2, v_4, and each of them has two neighbours of the same color, so none of them can be b-dominating, a contradiction.

Case 2: $d_3 = v_4$. Since v_1 is b-dominating, it has a neighbour of color 3, which can only be w_1, and a neighbour of color 4, which can only be v_5. Likewise v_2 has a neighbour of color 3, which can only be w_2, and a neighbour of color 4, which can only be v_3. But then v_4 has two neighbours of color 4, so it cannot be b-dominating, a contradiction. □

Theorem 2 ([6]) *Every d-regular graph G with girth $g(G) \geq 6$ has a b-coloring with $d+1$ colors.*

Proof. Let x be a vertex of G, and let $x_1, \ldots, x_d$ be its neighbors. For each $i = 1, \ldots, d$, let $N_i = N(x_i) \setminus \{x\}$. Then each N_i is a stable set, for otherwise G would contain a cycle of length three. Then any two N_i's are disjoint, for otherwise G would contain a cycle of length four, and there is no edge between them, for otherwise G would contain a cycle of length five. Thus we can obtain a coloring with $d+1$ colors $0, 1, \ldots, d$ as follows. Assign color 0 to x, and for $i = 1, \ldots, d$, assign color i to x_i, and assign to the vertices of N_i the colors from $\{1, \ldots, d\} \setminus \{i\}$, in a one-to-one fashion. Finally, color the remaining vertices in arbitrary order, assigning to each v a color from $\{0, 1, \ldots, d\}$ different from the colors already assigned to its neighbors. Clearly, we obtain a b-coloring that uses $d+1$ colors in which the vertices $x, x_1, \ldots, x_d$ are b-dominating. □

2 A new proof of El-Sahili and Kouider's theorem

El-Sahili and Kouider [2] proved the following theorem.

Theorem 3 ([2]) *If G is a d-regular graph with girth $g(G) \geq 5$ and G contains no C_6, then $b(G) = d+1$.*

We propose here a new proof, using a theorem of Gravier [4] on list coloring. Let L be a mapping that assigns to each vertex v of a graph G a set $L(v)$ of admissible colors. An L-coloring is a coloring c of the vertices of G such that $c(v) \in L(v)$ for every vertex v of G. If G admits an L-coloring it is said to be L-colorable. Let l be a mapping that assigns a positive integer $l(v)$ to each vertex v of G. Then G is said to be *l-list-colorable* if it admits an L-coloring for every L such that $|L(v)| \geq l(v)$ for every $v \in V$. Given an integer k, G is

k-list-colorable if it is l-list-colorable for the mapping $l(v) = k$, $\forall v \in V$. The *list-chromatic number* $\chi_L(G)$ is the smallest integer k such that G is k-list-colorable.

Theorem 4 **([4])** *Let G be a connected graph different from a complete graph and from an odd cycle. Then G is Δ-list-colorable, where Δ is the maximum degree in G.*

Proof of Theorem 3. If $d = 2$, then G is a disjoint union of cycles, all of length 5 or more. Then $b(G) = 3$. Indeed, in order to obtain a b-coloring with 3 colors, it suffices to give colors $1, 2, 3, 1, 2$ to five consecutive vertices in one cycle of G, and to color the rest of G with colors $1, 2, 3$. Now we assume that $d \geq 3$.

Let v be a vertex of G, with neighborhood set $N(v) = \{v_1, v_2, \ldots, v_d\}$, and let G_v be the subgraph of G induced by $(N(v_1) \cup \cdots \cup N(v_d)) \setminus \{v\}$. For $i = 1, \ldots, d$, set $E_i = \{xy \mid x, y \in N(v_i) \setminus \{v\}\}$. Then let H_v be the graph obtained from G_v by adding into its edge-set all the elements of $E_1 \cup \cdots \cup E_d$. We claim that:

$$\text{If } \chi_l(H_v) = d - 1, \text{ then } b(G) = d + 1. \tag{1}$$

To prove (1), we define a list assignment L on H_v as follows. If x is any vertex of H_v, we have $x \in N(v_i)$ for some neighbor v_i of v; then assign the list $L(x) = \{1, \ldots, d\} \setminus \{i\}$ to x. Since $\chi_l(H_v) = d - 1$, there is an L-coloring c of H_v. Extend c to a coloring of G as follows. Give color 0 to v; for $i = 1, \ldots, d$, give color i to the i-th neighbor v_i of v; then color the remaining vertices in arbitrary order, giving to each vertex z a color from $\{0, 1, \ldots, d\}$ different from the colors already assigned to its neighbors. Then we obtain a coloring of G with $d + 1$ colors, and clearly the vertices $v, v_1, \ldots, v_d$ are b-dominating. So (1) holds.

Now consider any vertex x in G_v. Then $x \in N(v_i)$ for some i, $1 \leq i \leq d$. Vertex x has no neighbor in N_i, for otherwise G would contain a cycle of length three. Vertex x cannot have two neighbors $y, z \in N_j$ $(j \neq i)$, for otherwise x, y, z, v_j would induce a cycle in G. Then x cannot have two neighbors $y \in N_j$, $z \in N_k$ (with i, j, k pairwise different), for otherwise x, y, z, v_j, v_k, v would induce a cycle in G. Thus x has at most one neighbor in G_v. It follows that the maximum degree in H_v is at most $1 + (d - 2) = d - 1$. Moreover, there cannot be two edges $xy, x'y'$ between N_i and N_j, for otherwise v_i, x, y, x', y', v_j would induce a cycle in G. Thus G_v has at most $d(d - 1)/2$ edges. Then H_v does not contain a complete graph K_d, because its largest cliques are induced by the sets $N(v_i) \setminus \{v\}$ and have size $d - 1$. Also, if $d - 1 = 2$, H_v does not contain an odd cycle, because when $d = 3$, the graph H_v has 6 vertices, so the only possible odd cycle is a C_5, and some vertex of the C_5 would have degree 2 in G_v. It follows from Theorem 4 and Claim (1) that $b(G) = d + 1$. $\square$

3 When the degree is small

Theorem 5 *Let G be a d-regular graph with girth $g(G) \geq 5$, different from the Petersen graph, and with $d \leq 6$. Then $b(G) = d + 1$.*

Proof. We distinguish one case for each value of the degree.

Case 1: $d = 1$. Then G is a matching and clearly $b(G) = 2 = d + 1$.

Case 2: $d = 2$. Then G is a disjoint union of cycles, all of length at least 5. We can obtain a b-coloring with 3 colors by assigning colors $1, 2, 3, 1, 2$ to five consecutive vertices in one cycle of G, and to color the rest of G with colors $1, 2, 3$ greedily. So $b(G) = 3$.

Now let $d \geq 3$. We may assume that G contains a cycle of length 5, for otherwise the result follows from Theorem 2. Therefore in Cases 3, 4, and 5, we let $x_1, \ldots, x_5$ be five vertices of G that induce a cycle C in this order.

Case 3: $d = 3$. For each $i = 1, \ldots, 5$, let u_i be the neighbor of x_i that is not in the cycle C. First suppose that the edge $u_i u_{i+2}$ exists for every $i = 1, \ldots, 5$. Then the vertices $x_1, \ldots, x_5, u_1, \ldots, u_5$ induce the Petersen graph and form one connected component of G. Since G itself is not the Petersen graph, it must have another component Z. In that case, give color 1 to x_1, x_4, color 2 to x_2, u_5, color 3 to x_3, u_1, and color 4 to x_5, u_2, u_3 and to some vertex z of Z, and give colors $1, 2, 3$ to the neighbors of z. So x_1, x_2, x_3, z are b-dominating vertices of colors respectively $1, 2, 3, 4$, and this coloring can be extended to a coloring of G with four colors in any greedy way.

Now we can assume, up to symmetry, that $u_1 u_3$ is not an edge of G. Let us make a b-coloring with 4 colors such that x_1, x_2, x_3, u_2 are b-dominating vertices of colors $1, 2, 3, 4$ respectively. To do this, we first give color 1 to x_1, x_4, color 2 to x_2, color 3 to x_3, x_5, and color 4 to u_1, u_2, u_3. Note that x_1, x_2, x_3 are b-dominating vertices of colors $1, 2, 3$ respectively. Now consider u_2. Let a, b be the two neighbors of u_2 different from x_2. (Possibly $\{a, b\} \cap \{u_4, u_5\} \neq \emptyset$.) Note that a and b are not adjacent to x_1, x_2, x_3, for otherwise G would contain a cycle of length 3 or 4. Moreover, and for the same reason, each of a, b is adjacent to at most one of x_4, x_5; and if each of them is adjacent to one of x_4, x_5 then it is not to the same vertex; in other words the edge set between $\{a, b\}$ and $\{x_4, x_5\}$ is a matching of size at most two. So it is possible to give color 1 to one of a, b and color 3 to the other without having two adjacent vertices of the same color. Now u_2 is a b-dominating vertex of color 4. Finally this coloring can be extended to a coloring of G with four colors in any greedy way.

Case 4: $d = 4$. For each $i = 1, \ldots, 5$, let A_i be the set of the two neighbors of x_i that are not in C. Since G contains no cycle of length 3 or 4, it is easy to see that:

$$A_i \text{ is a stable set;} \tag{2}$$

$$A_i \cap A_j = \emptyset \text{ if } i \neq j; \tag{3}$$

$$\text{There is no edge between } A_i \text{ and } A_{i+1}; \tag{4}$$

We can make a b-coloring of G with five colors such that $x_1, \ldots, x_5$ are b-dominating vertices of colors $1, \ldots, 5$ respectively, as follows. For each $i = 1, \ldots, 5$, assign color i to x_i and colors $i + 2$ and $i + 3$ (modulo 5) to the two vertices of A_i. Fact (4) and the fact that the two colors assigned to the vertices of A_i are different from the two colors assigned to A_{i+2} ensure that no two adjacent vertices in $A_1 \cup \cdots \cup A_5$ receive the same color. Thus all vertices of $A_1, \ldots, A_5$ have received a color, and each of $x_1, \ldots, x_5$ has neighbors of all colors other than its own. Finally, since the uncolored vertices have degree 4, we can color them successively with one of the five colors, in any greedy way. Thus we obtain a b-coloring of

G with five colors.

Case 5: $d = 5$. For each $i = 1, \ldots, 5$, let A_i be the set of the three neighbors of x_i that are not in C. Since G contains no cycle of length 3 or 4, it is easy to see that:

$$A_i \text{ is a stable set;} \tag{5}$$
$$A_i \cap A_j = \emptyset \text{ if } i \neq j; \tag{6}$$
$$\text{There is no edge between } A_i \text{ and } A_{i+1}; \tag{7}$$
$$\text{Every vertex different from } x_i \text{ has at most one neighbor in } A_i. \tag{8}$$

For each $i = 1, \ldots, 5$, we can find a neighbor s_i of x_i such that the set $\{s_1, \ldots, s_5\}$ is a stable set, as follows. Pick any $s_1 \in A_1$, then $s_3 \in A_3 \setminus N(s_1)$, $s_5 \in A_5 \setminus N(s_3)$, $s_2 \in A_2 \setminus N(s_5)$, and $s_4 \in A_4 \setminus (N(s_1) \cup N(s_2))$. Such vertices exist because of (7) and (8). It follows from this construction that the set $S_5 = \{s_1, \ldots, s_5\}$ is indeed a stable set. We rename vertex s_1 as x_6. For $i = 1, \ldots, 5$, let $B_i = A_i \setminus \{s_i\}$; so $|B_i| = 2$. Let $B_6 = N(x_6) \setminus \{x_1\}$; so $|B_6| = 4$. Since G contains no cycle of length 3 or 4, it is easy to see that:

$$B_6 \text{ is a stable set;} \tag{9}$$
$$B_6 \cap (B_1 \cup B_2 \cup B_5) = \emptyset; \tag{10}$$
$$|B_6 \cap B_i| \leq 1 \text{ for } i \in \{3, 4\}; \tag{11}$$
$$\text{There is no edge between } B_6 \text{ and } B_1; \tag{12}$$
$$\text{Every vertex different from } x_6 \text{ has at most one neighbor in } B_6. \tag{13}$$

We are going to make a b-coloring of G with six colors such that $x_1, \ldots, x_6$ will be b-dominating vertices of colors $1, \ldots, 6$ respectively. We start by assigning color i to x_i for each $i = 1, \ldots, 5$ and color 6 to the vertices of S_6. Now we must find a way to assign colors $i+2$ and $i+3$ (modulo 5) to the two vertices of B_i, for each $i = 1, \ldots, 5$, and colors $2, 3, 4, 5$ to the four vertices of B_6. We view this as a list-coloring problem, where each vertex of B_i ($i = 1, \ldots, 5$) has a list of allowed colors $L_i = \{i+2, i+3\}$ and each vertex of B_6 has a list of allowed colors $L_6 = \{2, 3, 4, 5\}$. During our coloring procedure, we will say that a vertex x *loses* a color j if this color must be removed from the list of allowed colors for x.

Recall from (11) that B_6 may have one common vertex with any of B_3, B_4. Let the vertices of B_6 be called a, b, c, d such that: if $B_6 \cap B_3 \neq \emptyset$, then a is the (unique) vertex in that intersection; and if $B_6 \cap B_4 \neq \emptyset$, then b is the (unique) vertex in that intersection. Assign colors $5, 2, 3, 4$ to a, b, c, d respectively.

By (7) there is no edge between B_i and B_{i+1} (modulo 5). Moreover, the sets of colors we want to assign to B_i and B_{i+2} are disjoint, so we can ignore the edges between these two sets. Therefore we can color the sets $B_1, \ldots, B_5$ independently from each other with no conflict between any two sets.

By (12), we can color the two vertices of B_1 with colors 3 and 4. Because of the assignment in B_6, and by (8), for each $j \in \{4, 5\}$ at most one vertex of B_2 loses color j, and that is a different vertex for each j. So it is possible to color the two vertices of B_2 with colors 4 and 5. The same holds for A_5 with colors 2 and 3. Now consider B_3. If $a \in B_3$, then a is a vertex of color 5 in B_3 and the remaining vertex of B_3 can be colored 1. If $a \notin B_3$

then one vertex of B_3 may lose color 5, but it is still possible to color the two vertices of B_3 with colors 1 and 5. The same holds for B_4 with colors 1, 2. Thus all vertices of $B_1, \ldots, B_6$ have received a color, and each of $x_1, \ldots, x_6$ has neighbors of all colors other than its own. Finally, since the uncolored vertices have degree 5, we can color them successively with one of the six colors, in any greedy way. Thus we obtain a b-coloring of G with six colors.

Case 6: $d = 6$. The proof here uses similar arguments as in the case $d = 5$, but the situation is more complicated. We can assume that G contains a cycle of length 6, for otherwise the result follows from Theorem 3. Let $x_1, \ldots, x_6$ be six vertices of G that induce a cycle in this order. For each $i = 1, \ldots, 6$, let $A_i = N(x_i) \setminus \{x_{i-1}, x_{i+1}\}$ (here indices are understood modulo 6); so $|A_i| = 4$. Since G contains no cycle of length 3 or 4, it is easy to see that:

$$A_i \text{ is a stable set;} \tag{14}$$

$$A_i \cap A_{i+1} = \emptyset \text{ and } A_i \cap A_{i+2} = \emptyset; \tag{15}$$

$$|A_i \cap A_{i+3}| \le 1; \tag{16}$$

$$\text{There is no edge between } A_i \text{ and } A_{i+1}; \tag{17}$$

$$\text{Every vertex different from } x_i \text{ has at most one neighbor in } A_i. \tag{18}$$

For each $i = 1, \ldots, 6$, we find a neighbor s_i of x_i such that the set $\{s_1, \ldots, s_6\}$ is a stable set, as follows:
- If $A_1 \cap A_4 \neq \emptyset$, let s_1 and s_4 be equal to the (unique) vertex in $A_1 \cap A_4$. If $A_1 \cap A_4 = \emptyset$, let s_1 be any vertex in A_1 and s_4 be any vertex in $A_4 \setminus N(s_1)$ (s_4 exists by (18)).
- If $A_3 \cap A_6 \neq \emptyset$, let s_3 and s_6 be equal to the (unique) vertex in $A_3 \cap A_6$. Note that, by (17), this vertex is not adjacent to the vertices s_1 and s_4 found previously. If $A_3 \cap A_6 = \emptyset$, let s_3 be any vertex in $A_3 \setminus N(s_1)$ and s_6 be any vertex in $A_6 \setminus (N(s_4) \cup N(s_3))$ (vertices s_3 and s_6 exist by (18)).
- If $A_5 \cap A_2 \neq \emptyset$, let s_5 and s_2 be equal to the (unique) vertex in $A_5 \cap A_2$. Note that, by (17), this vertex is not adjacent to any of the vertices s_1, s_3, s_4, s_6 found previously. If $A_5 \cap A_2 = \emptyset$, then, by (18), there are at least two vertices in $A_5 \setminus (N(s_1) \cup N(s_3))$ and at least two vertices in $A_2 \setminus (N(s_4) \cup N(s_6))$; and by (18) again, among these four vertices there are non-adjacent vertices $s_5 \in A_5$ and $s_2 \in A_2$.

It follows from this construction that the set $S_7 = \{s_1, \ldots, s_6\}$ is indeed a stable set. We rename vertex s_1 as x_7. For $i = 1, \ldots, 6$, let $B_i = A_i \setminus \{s_i\}$; so $|B_i| = 3$. Note that $B_1, \ldots, B_6$ are pairwise disjoint by (15) and the definition of S_7. Let $B_7 = N(x_7) \setminus \{x_1\}$; so $|B_7| = 5$. Since G contains no cycle of length 3 or 4, it is easy to see that:

$$B_7 \text{ is a stable set;} \tag{19}$$

$$B_7 \cap (B_1 \cup B_2 \cup B_6) = \emptyset; \tag{20}$$

$$|B_7 \cap B_i| \le 1 \text{ for } i \in \{3, 4, 5\}; \tag{21}$$

$$\text{There is no edge between } B_7 \text{ and } B_1; \tag{22}$$

$$\text{Every vertex different from } x_7 \text{ has at most one neighbor in } B_7. \tag{23}$$

An easy consequence of (18) and (23) is the following:

$$\text{If sets } X \subseteq B_i \text{ and } Y \subseteq B_j \text{ are such that } |X| > |Y|, \text{ then some vertex of } X \text{ has no neighbor in } Y. \tag{24}$$

We are going to make a b-coloring of G with seven colors such that $x_1, \ldots, x_7$ will be b-dominating vertices of colors $1, \ldots, 7$ respectively. We start by assigning color i to x_i for each $i = 1, \ldots, 6$ and color 7 to the vertices of S_7. Now we must find a way to assign colors $i+2, i+3, i+4$ (modulo 6) to the three vertices of B_i, for each $i = 1, \ldots, 6$, and colors $2, 3, 4, 5, 6$ to the five vertices of B_7. We view this as a list-coloring problem, where each vertex of B_i ($i = 1, \ldots, 6$) has a list of allowed colors $L_i = \{i+2, i+3, i+4\}$ and each vertex of B_7 has a list of allowed colors $L_7 = \{2, 3, 4, 5, 6\}$. During our coloring procedure, we will say that a vertex x *loses* a color j if this color must be removed from the list of allowed colors for x.

Recall from (21) that B_7 may have one common vertex with any of B_3, B_4, B_5. Up to symmetry, we may assume that $|B_7 \cap B_3| \geq |B_7 \cap B_5|$, in other words, if B_7 intersects one of B_3, B_5 then it intersects B_3. Define vertices a, b, c of B_7 as follows, where we distinguish two cases:

Case (i): $B_7 \cap B_4 \neq \emptyset$. Let a be the (unique) vertex in $B_7 \cap B_4$. Then, if $B_7 \cap B_3 \neq \emptyset$, let b be the (unique) vertex in $B_7 \cap B_3$ (note that b has no neighbor in B_4 by (17)); else, let b be a vertex in $B_7 \setminus \{a\}$ that has no neighbor in B_4 (such a vertex exists by (24)). Finally, if $B_7 \cap B_5 \neq \emptyset$, let c be the vertex in $B_7 \cap B_5$; else, let c be any vertex in $B_7 \setminus \{a, b\}$.

Case (ii): $B_7 \cap B_4 = \emptyset$. If $B_7 \cap B_3 \neq \emptyset$, let b be the (unique) vertex in this intersection (note that b has no neighbor in B_4 by (17)); else, let b be a vertex in B_7 that has no neighbor in B_4 (such a vertex exists by (24)). Then, if $B_7 \cap B_5 \neq \emptyset$, let c be the vertex in this intersection; else, let c be any vertex in $B_7 \setminus \{b\}$. Finally, let a be any vertex in $B_7 \setminus \{b, c\}$.

Note that in all cases vertices a, b, c are well-defined and different since B_3, B_4, B_5 are pairwise disjoint as mentioned above; and b has no neighbor in B_4. Then d is chosen as follows: If $b \in B_3$, $c \in B_5$, a has a neighbor $v_5 \in B_5$ and v_5 has a neighbor $v_3 \in B_3$, then choose $d \in B_7 \setminus \{a, b, c\}$ and not adjacent to v_3 (such a vertex exists by (24)); else let d be any vertex in $B_7 \setminus \{a, b, c\}$.

Finally let e be the remaining vertex of B_7. Assign colors $2, 6, 3, 5, 4$ to a, b, c, d, e respectively. Pick a vertex $f_2 \in B_2$ not adjacent to e and a vertex $f_6 \in B_6$ not adjacent to e or f_2; such vertices exist by (18). Assign color 4 to f_2 and f_6.

By (17) there is no edge between B_1 and $B_2 \cup B_6$. Moreover, the sets of colors we want to assign to B_1 and B_4 are disjoint, so we can ignore the edges between B_1 and B_4. The same as for B_1 holds for B_3 and for B_5. Therefore we can color $B_1 \cup B_3 \cup B_5$ and $B_2 \cup B_4 \cup B_6$ independently of each other with no conflict between the two sets.

Let us consider B_2, B_4, B_6. Because of the assignment in B_7, and by (18), for each $j \in \{5, 6\}$ at most one vertex of $B_2 \setminus \{f_2\}$ loses color j (a different vertex for each j). So it is possible to assign colors 5 and 6 to the two vertices of $B_2 \setminus \{f_2\}$. Likewise, for each $k \in \{2, 3\}$ at most one vertex of $B_6 \setminus \{f_6\}$ loses color k (a different vertex for each k), so it is possible to assign colors 2 and 3 to the two vertices of $B_6 \setminus \{f_6\}$. Call t the vertex of B_6 that receives color 2; so t is not adjacent to a. There remains to color the vertices of B_4. First suppose that $a \in B_4$ (case (i)). Then a is a vertex of color 2 in B_4 (recall that a, t are not adjacent), and the two vertices of $B_4 \setminus \{a\}$ lose color 2. Because of the assignment in $B_2 \cup B_6$, and by (18), at most one vertex of B_4 can lose a color (color 6), and by the choice of b no other vertex of B_4 can lose color 6. So it is possible to assign colors 1 and 6 to the two vertices of $B_4 \setminus \{a\}$. Now suppose that $a \notin B_4$ (case (ii)). Because of the assignment in $B_7 \cup B_2 \cup B_6$, and by (18), at most two vertices of B_4 lose color 2 and by the choice of b at most one loses color 6. So it is possible to assign colors $1, 2, 6$ to the three vertices of B_4. Thus, in either case all vertices of $B_2 \cup B_4 \cup B_6 \cup B_7$ have received a color, and each of x_2, x_4, x_6, x_7 has neighbors of all colors other than its own.

Now we deal with B_3, B_5. First suppose that $c \in B_5$, which implies $b \in B_3$ since $|B_7 \cap B_3| \geq |B_7 \cap B_5|$. Then the vertices of $B_3 \setminus \{b\}$ lose color 6 and the vertices of $B_5 \setminus \{c\}$ lose color 3. Because of the assignment in B_7, at most one vertex v_3 of $B_3 \setminus \{b\}$ loses a color (color 5) and at most one vertex v_5 of $B_5 \setminus \{c\}$ loses a color (color 2). We assign color 1 to v_3 and v_5. Note that, by the choice of d, we may assume that v_3 and v_5 are not adjacent. Then we assign color 5 to the third vertex of B_3 and color 2 to the third vertex of B_5. Now suppose that $c \notin B_5$. Because of the assignment in B_7, for each $j \in \{2, 3\}$ at most one vertex of B_5 loses color j (a different vertex for each j), so some vertex $w_5 \in B_5$ loses no color. If $b \in B_3$, then the vertices of $B_3 \setminus \{b\}$ lose color 6 and, because of the assignment in B_7, at most one vertex v_3 of $B_3 \setminus \{b\}$ loses a color (color 5). So we assign color 1 to v_3 and color 5 to the remaining vertex of B_3. Because of this assignment in B_3, at most one vertex of B_5 loses color 1. So it is possible to assign colors $1, 2, 3$ to the three vertices of B_5. If $b \notin B_3$, then for each $j \in \{5, 6\}$ at most one vertex of B_3 loses color j (a different vertex for each j), so some vertex $w_3 \in B_3$ loses no color. By (18), among the four vertices of $(B_3 \cup B_5) \setminus \{w_3, w_5\}$, there are two non-adjacent vertices, one in B_3 and one in B_5, to which we assign color 1. Then it is possible to assign color 5 and 6 to the remaining vertices of B_3 and colors 2 and 3 to the remaining vertices of B_5.

Now we deal with B_1. Recall that (19) holds. Because of the assignment in B_3 at most one vertex of B_1 loses a color (color 5) and because of the assignment in B_5 at most one vertex of B_1 (possibly the same vertex) loses a color (color 3). So it is possible to color the three vertices of B_1 with the colors $3, 4, 5$. Thus all vertices of $B_1 \cup B_3 \cup B_5$ have received a color, and each of x_1, x_3, x_5 has neighbors of all colors other than its own.

Finally, since the uncolored vertices have degree 6, we can color them successively with one of the seven colors, in any greedy way. Thus we obtain a b-coloring of G with seven colors. □

The proof above illustrates a technique which can probably not be extended to the general case. Indeed we tried to make a similar proof for graphs with $d = 7$, but the case analysis seems to become inextricable.

In conclusion we propose the following reformulation of El-Sahili and Kouider's question:

Conjecture 1 *Every d-regular graph with girth at least 5, different from the Petersen graph, has a b-coloring with $d + 1$ colors.*

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