Les cahiers Leibniz

# Path partitions, cycle covers and max k-chromatic subgraphs: some known, some forgotten, some proven conjectures and integer decomposition of coflows

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Path partitions, cycle covers and max $k$-chromatic subgraphs: some known, some forgotten, some proven conjectures and integer decomposition of coflows
(Lecture Note)

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**Abstract**

We first present pairs of assertions on path partitions in arbitrary digraphs, and on cycle covers in strongly connected graphs, connecting both objects to maximum $k$-chromatic subgraphs. We select analogies: statements that hold for path partitions and at least do not admit a counterexample for cycle covers, and vice versa.

In this pairing, the mate of some theorems is a conjecture. The analogy puts forward some new conjectures, some of which are converted to theorems:

– On the one hand, the assertions concerning cycle covers are almost all theorems, and form a round theory of a pyramid form: there is a general theorem implying the others. This is a minmax theorem on the maximum union of $k$ particular stable sets, and the minimum "value" of a cycle cover, where the definition of "value" depends on $k$.

The first presentation of the proof of this theorem (here in Haifa in 2006, and in Journal of Combinatorial Theory/B, 97 (4) (2007) 518-552) used network flows, polyhedral combinatorics – linear programming. We update that presentation by showing two different extremities: we provide a simplified polyhedral approach using Cameron and Edmonds' coflow polyhedra subject to a general argument of Baum and Trotter on integer decomposition; to honor Marty Golumbic's birthday and pure graph theory, in the talk we explain how this works in terms of direct algorithmic steps in the underlying graph.

– On the other hand, concerning path partitions, most of the assertions remain conjectures. Some simpler and weaker results are proved here using the relations between path partitions and cycle covers.

# Introduction

Partitioning the vertex-set of a graph by a minimum number of paths is one of the most natural problems concerning graphs. Minimizing the number of paths in such a partition contains the Hamiltonian Path problems both in the directed and undirected case.

For undirected graphs some variants involve matching theory some others the connectivity of graphs. Some results concern only particular classes of graphs. The only general result about minimum partitions in undirected graphs concern intersection graphs of paths in a tree, by Monma and Wei [27], a class later generalized by [19], [20].

For digraphs, a classical theorem of Gallai and Milgram *relates the problem to the stability number of a graph.* Path partitions in digraphs have been treated both with elegant graph theory, network flows, and polyhedral combinatorics, but have not yet revealed all of their secrets:

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Conjectures of Berge [4] and Linial [26] about the relation of maximum sets of vertices inducing a $k$-chromatic subgraph and particular path partitions resist through the decades. Hartman's excellent survey [22] witnesses of the variety of the methods that have been tried out with a lot of partial results but no breakthrough as far as the general conjectures are concerned. Some other conjectures are less well-known or have not yet been stated.

Led by analogies, we ask and answer in this talk more questions, and point at some connections.

Section 1 states analogous pairs of theorems on path partitions and cycle covers.

Section 2 presents the results concerning cycle covers deducing all from a general theorem.

Section 3 presents some results on path partitions, and some connections of these to cycle covers.

**Notation and Terminology**: Let $G = (V, E)$ a digraph. A *path* of a digraph is an ordered set $P = (v_1, \dots, v_{|P|})$ of vertices, all different, so that $v_i v_{i+1} \in E$ $(i = 1, \dots, |P| - 1)$. We will denote $\mathrm{ini}(P) := v_1$ the *initial* (first) vertex of a path and $\mathrm{ter}(P) := v_{|P|}$ the *terminal* (last) vertex of it. For us a path will be a vertex-set, that is, with an abuse of notation we will apply set-operations involving a path $P$, and in this case $P$ is just the set of its vertices. If the first and last points are equal it is called a *cycle* which also included one element sets (even if there is no incident loop). A *subpartition* is just a family of disjoint subsets of $V$.

For a subpartition $\mathcal{P}$, $R(\mathcal{P}) := V \setminus \cup\mathcal{P}$ will be called the *remainder* of $\mathcal{P}$. This complementation concerns the complement *with respect to the graph in which it is defined.* If we delete some vertices and the new vertex set is $V'$, for a subpartition $\mathcal{P}'$ of $V'$, $R(\mathcal{P}')$ is defined as $R(\mathcal{P}') := V' \setminus \cup\mathcal{P}'$. We apply this notation only when the vertex-set to which we apply it is clear. A subpartition of paths is a *path partition* if and only if $R(\mathcal{P}) = \emptyset$.

If we do not say otherwise, $G = (V, E)$ is a digraph, $n := |V|$ and $m := |E|$.

A *stable* set is a subset of vertices that does not induce any edge. The maximum size of a stable set of a graph $G$ is denoted by $\alpha = \alpha(G)$. The *chromatic number* is the minimum size of a partition of $V$ into stable sets, and is denoted by $\chi = \chi(G)$. The maximum size of a $k$-chromatic induced subgraph (equivalently, the union of $k$ stable sets) is denoted by $\alpha_k$; $\alpha_1 = \alpha$. The *chromatic number* is the minimum size of a partition of $V$ into stable sets, and is denoted by $\chi = \chi(G)$. The minimum size of a partition into paths is denoted by $\pi = \pi(G)$, and the minimum cover by cycles is $\zeta = \zeta(G)$. By convention $\{v\}$ is also a cycle for all $v \in V$ (and it is of course a path too). So $\pi, \zeta \le n$. The number of vertices of the longest path is denoted by $\lambda = \lambda(G)$.

Two families of sets are called *orthogonal* if taking one set of each, the intersection of the two sets is always 1. A family is said to *cover* a set, if the union of its members contains the set.

The subgraph induced by a set $X \subseteq V$ will be denoted by $G(X)$, just replaced by $X$ to avoid double parentheses, for instance $\alpha(X) := \alpha(G(X))$; "strongly connected" will sometimes be replaced by *strong*.

# 1 Pairs of Assertions

## 1.1 Tournaments ($\alpha = 1$)

A *tournament* is an oriented complete graph.

**Theorem 1.1** (Rédei [28]) *Let $G$ be a tournament. Then it has a Hamiltonian Path.*

**Theorem 1.2** (Camion [12]) *Let $G$ be a strong tournament. Then it has a Hamiltonian Cycle.*

We state "light" and "heavy" versions of some assertions. The former refers to inequalities that generalize the nontrivial inequalities of minmax theorems (of Dilworth's, of Greene-Kleitman's or of some more recent ones), and the latter generalize complementary slackness:

### 1.2 Stability ($k = 1$, light)

**Theorem 1.3** (Gallai, Milgram [17]) *Let $G$ be an arbitrary digraph. Then $\alpha \geq \pi$.*

**Theorem 1.4** (Bessy, Thomassé [7], Gallai's conjecture [16]) *Let $G$ be strong . Then $\alpha \geq \zeta$.*

Specializing these to acyclic transitive digraphs both imply the nontrivial part of Dilworth's theorem stating equality in the former theorem for acyclic transitive digraphs (posets), see for instance [33]. To deduce it from the latter theorem, we first have to make an acyclic transitive digraph strongly connected. This can be done for instance by adding a "supersource" and joining it to all the vertices of 0 indegree (sources), adding a "supersink" and joining all the vertices of 0 outdegree (sinks) to it, and adding an arc from the supersink to the supersource.

### 1.3 Stability ($k = 1$, heavy)

Any proof of the Gallai-Milgram theorem obviously provides the following:

**Theorem 1.5** *Let $G$ be an arbitrary digraph. Then there exists a path partition $\mathcal{P}$ of $G$ and a stable set orthogonal to $\mathcal{P}$.*

Since Theorem 1.4 is the weakening of a min-max theorem, the condition of equality ("complementary slackness") easily implies:

**Theorem 1.6** *Let $G$ be a strong digraph. Then there exists a cycle cover $\mathcal{C}$ of $G$ and a stable set $S$ orthogonal to $\mathcal{C}$, where each element of $S$ is covered by exactly one member of $\mathcal{C}$.*

Of course these statements can also be specialized to Dilworth's theorem, with the same reduction as before.

### 1.4 Coloring ($k = \lambda$, light)

**Theorem 1.7** (Gallai, Roy [18], [29]) *Let $G$ be an arbitrary digraph. Then there exists a path of size at least $\chi$.*

**Theorem 1.8** (Bondy [8]) *Let $G$ be a strong digraph. Then there exists a cycle of size at least $\chi$.*

### 1.5 Coloring ($k = \lambda$, heavy)

Any proof of the Gallai-Roy theorem obviously provides the following:

**Theorem 1.9** *[8] Let $G$ be an arbitrary digraph. Then for any longest path there exists a colouring whose color classes are orthogonal to the path.*

Bondy's theorem does correspond to an LP duality theorem, but the vertices are not stable sets and cycles ; the corresponding polyhedron does have fractional vertices [31], and it is not evident how it would imply an analogous theorem for the longest cycle in a strongly connected graph. Nevertheless, this heavy version of Bondy's theorem is true, and this is actually what Bondy proved:

**Theorem 1.10** *[8] Let $G$ be a strong digraph. Then there exists a cycle and a colouring so that the color classes are orthogonal to the cycle.*

Here is a somewhat different structural sharpening of the Gallai-Roy theorem:

**Conjecture 1** (Laborde, Payan, Xuong [25]) *Let $G$ be an arbitrary digraph. Then there exists a stable set in $G$ that meets every longest path.*

Could this be true replacing "longest path" by "longest cycle" in strongly connected graphs ?

## 1.6 General (light)

**Conjecture 2** (Linial [26]) *Let $G$ be a digraph. Then $\alpha_k \geq \min_{X \subseteq V}\{|X| + k\pi(V \setminus X)\}$.*

**Theorem 1.11** (Sebő [31]) *Let $G$ be a strong digraph. Then $\alpha_k \geq \min_{X \subseteq V}\{|X| + k\zeta(V \setminus X)\}$.*

**Corollary 1.1** *(*Greene-Kleitman [21]) Let $G$ be a transitive acyclic digraph. Then $\alpha_k = \min_{X \subseteq V}\{|X| + k\zeta(V \setminus X)\}$.

Indeed, for transitive acyclic digraphs " $\leq$ " is easy, and to prove the nontrivial inequality of the Greene-Kleitman theorem, the reduction of Subsection 1.2 to strongly connected graphs works again. So the corollary indeed follows from the preceding theorem. Note that the right hand side of the Greene-Kleitman theorem or of Linial's conjecture is usually written as min $\{\sum_{P \in \mathcal{P}} \min\{|P|, k\} : \mathcal{P}$ *is a path* and this sum is called the *$k$-norm* of $\mathcal{P}$.

## 1.7 General (heavy)

The following is a simple already unknown version of Berge's conjecture. It implies Linial's conjecture. (The two conjectures have been stated independently.)

**Conjecture 3** (Berge [4]) *Let $G$ be a digraph, and $k \in \mathbb{N}$, $k \geq 1$. Then there exists $X \subseteq V$, a path partition $\mathcal{P}$ of $V \setminus X$, and $k$ disjoint stable sets covering $X$ and orthogonal to $\mathcal{P}$.*

There are several options for replacing "there exists" by "for all" in this conjecture. However, we can prove the strongly connected version only in the following form:

**Theorem 1.12** (Sebő [31]) *Let $G$ be a strong digraph, and $k \in \mathbb{N}$, $k \geq 1$. Then there exists $X \subseteq V$, a cycle cover $\mathcal{C}$ of $V \setminus X$, and $k$ disjoint stable sets covering $X$, all orthogonal to $\mathcal{C}$. Furthermore, each element of the stable sets is covered by at most one member of $\mathcal{C}$.*

This last theorem easily implies all that has been previously stated about covers in strongly connected graphs:

It shows $k$ stable sets whose union $U$ satisfies $|U| = |X| + k|\mathcal{C}|$ for some $X \subseteq V$ and cycle cover $\mathcal{C}$ of $V \setminus X$. Theorem 1.11 follows since $\alpha_k \geq |U|$. Theorem 1.12 and Theorem 1.11 are central in our presentation. We show here how they can be proved through the integer decomposition property of coflow polyhedra.

Theorem 1.8 follows from Theorem 1.12 because choosing $k$ to be the size of the longest cycle, $|X| + k|\mathcal{C}| \geq |X| + |\cup\mathcal{C}| \geq n$, so the union of the $k$ disjoint stable sets provided by Theorem 1.12 is at least $n$. So $G$ can be colored by $k$ colors.

The $k = 1$ special case of Theorem 1.12 is Theorem 1.6, itself implying Theorem 1.4. Indeed, in this case the elements in $X$ can be replaced by 1-element cycles.

Theorem 1.12 will, in turn, be proved in Section 2.3.

## 2 Cycle Covers

The ultimate goal of this section is to prove the second theorem of each subsection of Section 1. They have already been proved from Theorem 1.12, and here we will prove this latter. There are some interesting tools on the way, and they will lead us further: the integrality and integer decomposition property of *coflow* polyhedra, and the *coherent orders* of Knuth, Bessy and Thomassé. For the notations and basic notions from polyhedral combinatorics (including TDI, integer decomposition, etc.) we refer in this extended abstract to [32], [33]. The talk will be self-contained.

### 2.1 Coflows and Integer Decomposition

We wish to introduce here a ready to use helpful treatment of node-capacitated circulation problems. The idea is well-known: node-capacities can be reduced to edge-capacities by splitting each vertex $v$ into two copies, an in-copy $v_{\text{in}}$ and an out-copy $v_{\text{out}}$ and adding the arc $v_{\text{in}}v_{\text{out}}$ with the given vertex-capacities (possibly lower and upper), see [32], [11]. It is less well-known that relevant cycle-cover or cycle packing problems arise in this way and have useful properties, such as box total dual integrality, primal integrality if the parameters are integers [11], furthermore integer decomposition (below). This elegant tool defined by Cameron and Edmonds is defined as follows:

The *coflow system of inequalities* $\mathcal{Q}(G, a, b, c)$, where $G = (V, E)$ is a digraph, $a, b : V(G) \to \mathbb{Z}$, $c : E \to \mathbb{Z}$ is the following system in $n := |V|$ variables $x_v$ $(v \in V)$:

$$x(V_C) \leq c(E_C) \text{ for every cycle } C \text{ with vertex-set } V_C \text{ and edge-set } E_C \ ,$$

$$a \leq x \leq b.$$

The set of points $x \in \mathbb{R}^V$ satisfying the coflow inequalities $\mathcal{Q}(G, a, b, c)$ is called the *coflow polyhedron*, and is denoted by $Q(G, a, b, c)$. The *coflow (primal) problem* $P(G, a, b, c, w)$, where $G, a, b, c$ are as before, and $w : V(G) \to \mathbb{Z}$, is the following:

$$\max\{w^\top x : x \in Q(G, a, b, c)\}$$

$D(G, a, b, c, w)$ will denote the dual problem, and $\text{opt}(G, a, b, c, w)$ the common optimum of the primal and the dual (which can also be infinite).

**Theorem 2.1** *(Coflow Theorem [9],[11]) Any system of coflow inequalities $\mathcal{Q}(G, a, b, c)$ is TDI.*

**Sketch of the Proof**: For every $w : V(G) \to \mathbb{Z}$ the dual problem $D(G, a, b, c, w)$ is a problem of covering vertices by cycles which is a flow problem with $w$ as lower capacities on the vertices. Apply the mentioned splitting up of vertices to have edge-capacities in the usual way instead of vertex-capacities. Then it is easy to see that $D(G, a, b, c, w)$ becomes a flow problem – more precisely a query about the existence of a lower capacitated circulation. (While $P(G, a, b, c, w)$ can be stated as a problem of maximizing vertex-weights that can be subtracted from the weights of all edges entering each vertex, without creating negative cycles.) From Hoffman's circulation theorem [33] one gets then the following lemma that we state explicitly for later use, and which finishes the proof of the theorem:

**Lemma 2.1** *For every coflow polyhedron $Q(G, a, b, c)$ there exists a totally unimodular matrix $A$ with $n' > n$ rows, $m' > m$ columns and an integer vector $c' \in \mathbb{Z}^{m'}$ so that*

$$Q(G, a, b, c) = \{(y_1, \ldots, y_n) : yA \leq c'\}, \tag{1}$$

*and for all $w : V(G) \to \mathbb{Z}$ there exists $b' \in \mathbb{Z}^{m'}$ so that*

$$\{(x_1, \ldots, x_m) : \ x \in \mathbb{N}^m \text{ minimizes } c'^\top x \text{ subject to } Ax = b',\ x \geq 0\} \tag{2}$$

*is a set of circulations in $G$, and the set of cycle-decompositions of this set of circulations is equal to the set of optimal solutions of $D(G, a, b, c, w)$.*

A cycle-decomposition of the circulation $f$ is a set of cycles such that $f$ is the sum of their characteristic vectors (with multiplicities).

The second part of the Lemma immediately implies the coflow theorem. $\square$

It has been noticed in [11] that coflow polyhedra are projections of integer polyhedra, which is the "integer decomposition property" for $k = 1$. [31] has missed coflows – it applies directly flows in each special case: $c$ takes there only two different values in all of these– 0 and $k \in \mathbb{N}$, which makes the proofs and algorithms simpler. However, *the integer decomposition property of $Q(G, a, b, c)$ is also proved for those special cases, which is crucial for proving Theorem 1.11 and 1.12.* This enables the inclusion of *unions of vertices of* $0 - 1$ *coflow polyhedra, establishing that these also form coflow polyhedra, analogously with a similar matroid property.* In [13], the special case of the coflow theorem which implies Bessy and Thomassé's theorem (Theorem 3.3) is restated and proved directly in terms of Totally Unimodular matrices. The presentation of $D(G, a, b, c, w)$ in the latter is probably the most convenient to use for checking the Lemma.

It is unfortunate that these special cases were proved one by one in these recent papers, without knowing about coflows. Several colleagues advised a unified treatment like this - Attila Bernáth made a very concrete suggestion; then I learned about coflows from Irith Hartman, but the non-trivial graph theoretic proof of the integer decomposition property in particular cases – where the tree is hiding the forest – persisted. I have realized only recently that this property can be proved for general coflows in a much simpler way than in [31], confirming the role of coflows:

**Theorem 2.2** *(Coflow ID) Coflow polyhedra have the integer decomposition property.*

**Proof**: First we mimic the proof of the easy part of a result of Baum and Trotter [3], [33]:

**Claim 1**: If $A$ is totally unimodular, $\{y : yA \leq c\}$ has the integer decomposition property.

Indeed, suppose $\bar{y}A \leq kc$. We have to show an integer vector $y_k$, $y_kA \leq c$ for which

$$(\bar{y} - y_k)A \leq (k-1)c.$$

Then the statement follows by induction on $k$. We have to find a linear solution to

$$\bar{y}A - (k-1)c \leq y_kA \leq c,$$

where $\bar{y}$, $A$, $k$, $c$ are fixed and the entries of $y_k$ are the variables. This system of linear inequalities has a solution, since $y_k := (1/k)\bar{y}$ is a solution. Since $A$ is unimodular, then it also has an integer solution, and the claim is proved.

**Claim 2**: If $Q \subseteq \mathbb{R}^{n'}$, $Q = \{y : yA \leq c\}$ where $A$ is a totally unimodular matrix, and $n < n'$, then

$$P := \{(y_1, \ldots, y_n) : y \in Q\}$$

has the integer decomposition property.

First note that $Q$ does have the integer decomposition property by Claim 1.

Let now $x \in kP \cap \mathbb{Z}^n$. Then $x/k \in P$, and by definition $x/k = (y_1, \ldots, y_n)$ for some $y \in Q$. So $ky \in kQ$, and the first $n$ entries of $ky$ are the entries of $x$: $ky = (x, x')$. Choose $x'$ to have a maximum number of integer coordinates. We show that it is an integer vector.

Indeed, $x'$ is a feasible solution of the equation $xB + x'A' \le c$, where $B$ is the matrix formed by the first $n$ rows of $A$, and $A'$ is the rest. Since $A$ is totally unimodular, $A'$ is also totally unimodular, so the equation

$$y'A' \le c - xB,$$

where $c$, $x$, $B$ are fixed and $y'$ is variable, also has an integer solution (it does have a feasible solution $x'$). Since $x' = y'$ is a possible choice, $x'$ and therefore $ky$ are indeed integer vectors. Consequently, because of the integer decomposition property of $Q$:

$$ky = y^1 + \ldots + y^k,\ y^i \in Q,\ \text{ and } y^i \text{ is an integer vector.}$$

Letting $x^i$ be the vector formed by the first $n$ entries of $y^i$, we get

$$x = x^1 + \ldots + x^k,\ x^i \in P \cap \mathbb{Z}^n \quad (i = 1, \ldots k),$$

finishing the proof of the claim.

To finish the proof of the Theorem note that by the Lemma, coflow polyhedra are of the form of the condition of Claim 2, therefore they have the integer decomposition property. □

## 2.2 Coherence

Graph theory courses characterize strongly connected graphs with the existence of an "ear decomposition" [33, Theorem 6.9]. Knuth's characterization is then at hand, and provides considerably more information:

**Theorem 2.3** *(Knuth) Let $G = (V, E)$ be a strong digraph. Then for every $v \in V$ there exists an order $v_1, \ldots, v_n$ on $V$ such that $v_1 = v$, and*

*(i) Every $e \in E$ is contained in a cycle $C$ with at most one backward arc.*

*(ii) Every $v \in V$ can be reached from $v_1$ using only forward arcs.*

A *backward arc* (with respect to a given order of the vertices) is an arc $v_iv_j \in E$, $i > j$, the other arcs are *forward arcs.*

Bessy and Thomassé [7] found the relevant part (i) independently, and developed it as a key to their proof of Gallai's conjecture Theorem 1.4. (A second ingredient was Dilworth's theorem that has been traded for circulations in [31]. The latter solution has been extended to a proof of Theorem 2.4, which is updated here.) Following them we call an order satisfying (i) *coherent.* They proved that every strong digraph has a coherent order. The equivalence of this fact with (i) in Knuth's theorem has been realized by Iwata and Matsuda [23].

Four simple proofs of the existence of coherent orders in strongly connected graphs, each providing its own insight, can be found in [24], [7], [31], [23].

Knuth proved this theorem as an application of his "Wheels within Wheels" theorem [24]. Iwata and Matsuda found Knuth's theorem in the archives, and proved it shortly and constructively using the ear decomposition of strongly connected graphs providing a measurable computational progress as well: it takes $\mathrm{O}(nm)$ time to construct the order in Theorem 2.3, whereas a construction in [31] based on different ideas takes $\mathrm{O}(n^2m^2)$ time.

Fixing an order, the *index* (or winding) $\text{ind}(C)$ of a cycle $C$ is the number of its backward arcs, except for cycles $\{v\}$ ($v \in V$) for which we define $\text{ind}(\{v\}) = 1$ (like if it had one "backward loop"). If $\mathcal{C}$ is a set of cycles

$$\text{ind}(\mathcal{C}) := \sum_{C \in \mathcal{C}} \text{ind}(C).$$

For any cycle $C$ in any graph with any order, $\text{ind}(C) \geq 1$, so for any set of cycles $\mathcal{C}$, we have $\text{ind}(\mathcal{C}) = |\mathcal{C}|$.

### 2.3 Topping

At the end of Section 1 we deduced the Greene-Kleitman theorem, and well-known results on cycle-covers, from Theorem 1.12. It is now the turn of Theorem 1.12 itself, completed by the following topping:

Given a graph $G = (V, E)$ with an order on the vertex set, let us call a set $S$ satisfying

$$\text{(COMB)} \qquad |S \cap C| \leq \text{ind}(C), (i = 1, \dots k).$$

a *cyclic stable set*. (This notion is equivalent to a geometric notion of Bessy and Thomassé [7]. The equivalence is proved in [31, (5)].)

The only thing we need here about cyclic stable sets though is that they *are indeed stable sets provided $G = (V, E)$ with the given order is coherent*. This is true, because by coherence every arc $e = ab \in E$ ($a, b \in V$) is contained in a cycle $C$ with $\text{ind}(C) = 1$, so we get for the sets $S$ satisfying (COMB): $|S \cap \{a, b\}| \leq |S \cap C| = 1$. So for every arc $ab \in E$, $S$ can contain at most one of $a$ and $b$.

**Theorem 2.4** *([31] Theorem 3.1) Let $G$ be a strong digraph given with a coherent order.*

$$\max\{|S_1 \cup \ldots \cup S_k| : \ S_i \ \textit{(i=1,\dots k) is a cyclic stable set}\ \} =$$

$$= \min\{|X| + k\,\text{ind}(\mathcal{C}) : \ X \subseteq V, \ \mathcal{C} \textit{ is a set of cycles that covers } V \setminus X\}.$$

**Proof**: Let $G = (V, E)$ be strong and $k \in \mathbb{Z}$. Apply Theorem 2.3 to $G$ and fix the coherent order it provides. Let $B$ be the set of backward arcs, and define $c_{B,k}(e) := k$ if $e \in B$, and 0 otherwise $a := 0 \in \mathbb{Z}^n$, $b := w := 1 \in \mathbb{Z}^n$ (constant 0 and constant 1 vectors, that we will simply denote by 0 and 1). Then $\mathcal{Q}(G, 0, 1, c_{B,k})$ is the following system:

$$(kBT) \qquad x(V_C) \leq k\,\text{ind}(C) \text{ for every cycle } C \text{ with vertex-set } V_C, \ 0 \leq x \leq 1.$$

**Claim 1**: The optimum of $P(G, 0, 1, c_{B,k}, 1)$ is $\max|S_1 \cup \ldots \cup S_k|$, $S_i \subseteq V$ ($i = 1, \dots k$) satisfies (COMB).

Such a union defines a primal solution, so the optimum is at least this quantity. To prove the equality, we show that the optimum $x_{\text{opt}}$ of $P(G, 0, 1, c_{B,k}, 1)$ can be written in this form. Note $Q(G, 0, \infty, c_{B,k}) = kQ(G, 0, \infty, c_{B,1})$, so $P(G, 0, 1, c_{B,k}, 1)$ is the problem

$$\max x_1 + \ldots + x_n, \text{ subject to } x \in kQ(G, 0, \infty, c_{B,1}), \quad x \leq 1.$$

Because of Theorem 2.2 applied to $Q(G, 0, \infty, c_{B,1})$:

$$x_{\text{opt}} = x^1 + \ldots + x^k, \ x^i \in Q(G, 0, \infty, c_{B,1}) \text{ for all } i = 1, \dots k,$$

and because of $x_{\text{opt}} \in \{0,1\}^n$ and $x^i \geq 0\,(i = 1, \dots k)$ *we have* $x^i \in \{0,1\}^n$, that is, $x_i$ is the incidence vector of a set $S_i$ satisfying (COMB).

**Claim 2**: The optimum of $D(G, 0, 1, c_{B,k}, 1)$ is

$$\min\{|X| + k\,\mathrm{ind}(\mathcal{C}) : X \subseteq V,\ \mathcal{C} \text{ is a set of cycles that covers } V \setminus X\}.$$

Indeed, (kBT) is a TDI system (Theorem 2.1), and therefore the dual optimum is a $0-1$ vector. Letting for a given dual solution $X := \{v \in V :$ the dual variable for $v$ is 1 $\}$, the dual optimum of (kBT) is as claimed.

We have arrived at the end of the proof now: by the duality theorem of linear programming $\mathrm{OPT}(G, 0, 1, c_{B,k}, 1)$ is equal to both the quantities in Claim 1 and Claim 2, and by (COMB) the sets $S_i$ $(i = 1, \dots, k)$ are all cyclic stable sets. $\square$

*Theorem 1.11 is an immediate corollary* since $\mathrm{ind}(\mathcal{C}) \geq |\mathcal{C}|$.

The stable sets $S_i$ $(i = 1, \dots k)$ and the set of cycles $\mathcal{C}$ provided by the theorem satisfy by complementary slackness (get it directly from the equalities of the theorem or in its proof) : $|S_i \cap C| = \mathrm{ind}(C) \geq 1$ $(i = 1, \dots k)$, $C \in \mathcal{C}$, so we can delete from each $S_i$ all but one of the elements of $S_i \cap C$, *finishing the proof of Theorem 1.12 as well.*

The original proof of theorems 2.4, 1.11, 1.12 was quite tedious – the integer decomposition property was proved through a complicated graph theory argument using potentials (arriving at a geometric surplus though). Theorem 2.2 provides a shorter way. (Which can also be converted into an algorithm.) Theorem 2.4 is actually the most general result we can prove for the union of $k$ stable sets. It is similar to sums of matroids.

Let us finally deduce from Theorem 2.4 the two fundamental results of Bessy and Thomassé [7] originally proved with two entirely different methods. They are both minmax theorems, so "structural versions" follow by complementary slackness.

**Corollary 2.1** *([7] Theorem 1) Let $G$ be a strong digraph given with a coherent order.*

$$\max\{|S| : S \textit{ is a cyclic stable set }\} = \min\{\mathrm{ind}(\mathcal{C}) : \mathcal{C} \textit{ covers } V\}.$$

**Proof** : Apply Theorem 2.4 to $k = 1$, noting that the one-element subsets of $X$ can be added in this case to $\mathcal{C}$. $\square$

For $k = 1$ the integer decomposition actually becomes simply flow integrality and we get back the simple proof of Theorem 1.6 in the introduction of [31] (Subsection 0.3).

**Corollary 2.2** *([7], combination of Lemma 3 and Theorem 3) The minimum of $k$ such that $G$ can be colored with $k$ cyclic stable sets is equal to the maximum of $\lceil |C|/\,\mathrm{ind}(C)\rceil$ over all cycles of $C$.*

**Proof** : Apply Theorem 2.4 to $k := \max\{\lceil |C|/\,\mathrm{ind}(C)\rceil : C$ is a cycle of $G$ $\}$. Then $k\,\mathrm{ind}(C) \geq |C|$ for every cycle, and therefore the right hand side in Theorem 2.4 is $n$. $\square$

These two corollaries showed the way: they are the two ice-cream balls, the theorem is the topping.

TDI and Integer Decomposition are just tools for finding the proof, and are not inherent fatalities here: they can be substituted by elementary steps. In the spirit of Marty Golumbic's birthday, and for an efficient continuation of the Israeli-French collaboration project, in the talk I wish to sketch a pure graph theory proof.

## 3 Path Partitions

Let us call a set $X$ for which the minimum in Conjecture 2 is reached a *minimizer*. Berge actually stated his conjecture for all minimizers and minimum path partitions of $V \setminus X$. We will prove this version in the following cases:

### 3.1 Long Paths

**Theorem 3.1** *Let $G = (V, E)$ be a digraph, $k, m \in I\!N$ and $\mathcal{P} = \{P_1, \dots, P_m\}$ a path partition. Then there exists*

*(i) either $k$ disjoint stable sets orthogonal to $\mathcal{P}$,*

*(ii) or a subpartition $\mathcal{Q} = \{Q_1, \dots, Q_{m-1}\}$ of paths s.t. $\mathrm{ini}(\mathcal{Q}) \subseteq \mathrm{ini}(\mathcal{P})$, $\mathrm{ter}(\mathcal{Q}) \subseteq \mathrm{ter}(\mathcal{P})$, and*

$$|R(\mathcal{Q})| \leq k - 1.$$

**Proof :** We prove the statement by induction on $n := |V|$. Suppose it holds for all $n' < n$ with all values of $m$ and $k$, and prove it for $G$.

We can suppose that $|P| \geq k$ for all $P \in \mathcal{P}$, because if say $|P_m| < k$, define $Q_i := P_i$ for all $i = 1, \dots, m-1$. We see that (ii) holds: $|V \setminus (Q_1 \cup \dots \cup Q_{m-1})| = |P_m| \leq k - 1$.

Let $a_i := \mathrm{ini}(P_i)$, and let $a'_i$ be the second vertex of $P_i$, and $P'_i := P_i \setminus \{a_i\}$, that is, $\mathrm{ini}(P'_i) = a'_i$ $(i = 1, \dots, m)$.

We distinguish now two cases:

**Case 1:** $\mathrm{ini}(\mathcal{P})$ is not a stable set, that is, say $a_1 a_2 \in E$.

If $|P_1| = k$, we can replace $\mathcal{P}$ by $\mathcal{P} \setminus \{P_1\}$, and add $a_1$ to $P_2$ as first vertex: we see that (ii) holds then.

So suppose $|P_1| \geq k + 1$, and apply the induction hypothesis to $G - a_1$ and the same path partition restricted to $V \setminus \{a_1\}$, that is, with the only change of replacing $P_1$ by $P_1 \setminus \{a_1\}$. If now (i) holds, then, using also that $|P_1 \setminus \{a_1\}| \geq k$, (i) also holds for $G$. So suppose (ii) holds for $G - a_1$, and let $\mathcal{Q}' = \{Q'_1, \dots, Q'_{m-1}\}$ be the path partition satisfying (ii). Since $\mathrm{ini}(\mathcal{Q}')$ is an $m-1$ element subset of $\{a'_1, a_2, \dots, a_m\}$, it contains a path $Q'_1$ with $\mathrm{ini}(Q'_1) = a'_1$ or $\mathrm{ini}(Q'_1) = a_2$. Adding $a_1$ as a first vertex to $Q'_1$ we get a path partition of $G$ that satisfies (ii).

**Case 2:** $\mathrm{ini}(\mathcal{P})$ is a stable set.

Apply the statement to $G' := G - \mathrm{ini}(\mathcal{P})$, $k' := k - 1$.

If then (i) holds, then adding the stable set $\mathrm{ini}(\mathcal{Q})$ to the provided $k-1$ stable sets, we get that (i) holds for $G$ and $\mathcal{P}$ with parameter $k$.

Otherwise alternative (ii) holds for $G' = (V', E')$ with $\mathcal{P}'$ and $k'$, that is, we have a subpartition of paths $\mathcal{Q}' = \{Q'_1, \dots, Q'_{m-1}\}$, in $G'$ such that $\mathrm{ini}(\mathcal{Q}') \subseteq \mathrm{ini}(\mathcal{P}')$ (and $\mathrm{ter}(\mathcal{Q}') \subseteq \mathrm{ter}(\mathcal{P}') = \mathrm{ter}(\mathcal{P})$), that is, with an appropriate choice of the notation $\mathrm{ini}(\mathcal{Q}') = \{a'_1, \dots, a'_{m-1}\}$, furthermore $\mathrm{ini}(Q'_i) = \{a'_i\}$ for all $i = 1, \dots, m-1$, and

$$|R(\mathcal{Q}')| = |V' \setminus (Q'_1 \cup \dots \cup Q'_{m-1})| \leq k - 2.$$

Define now $Q_i := Q'_i \cup a_i$ $(i = 1, \dots, m-1)$. Clearly, $\mathcal{Q} := \{Q_1, \dots, Q_{m-1}\}$ is a subpartition of paths in $G$, and

$$|R(\mathcal{Q})| = |V \setminus (Q_1 \cup \dots \cup Q_{m-1})| = |V' \setminus (Q'_1 \cup \dots \cup Q'_{m-1})| + 1 \leq k - 1,$$

since $V \setminus (Q_1 \cup \ldots \cup Q_{m-1}) = (V' \setminus (Q'_1 \cup \ldots \cup Q'_{m-1})) \cup \{a_m\}$, proving that alternative (ii) holds for $G$ with $\mathcal{P}$ and $k$. $\square$

This implies Berge's conjecture when there exists an empty minimizer, which turns out to be equivalent to a result of Aharoni, Hartman and Hoffman [1]. Their proof is based on improving paths, and probably implies all the claims of the Theorem, in a more involved way though.

**Corollary 3.1** *[1] If $G$ is a digraph and $\mathcal{P}$ is a path partition where $k|\mathcal{P}| = \min\{|X| + k\pi(V \setminus X) : X \subset V\}$, then there exist $k$ disjoint stable sets orthogonal to $\mathcal{P}$.*

## 3.2 Acyclic digraphs

For acyclic digraphs Berge's and Linial's conjectures are consequences of Theorem 2.2 on the lines of, and more simply than the proof of Theorem 2.4, but I do not see how to deduce them directly from the statement of Theorem 2.4. The results have been proved in [1], [2], [10], [14], [26], [30]. Let us show how Theorem 2.2 replaces all the difficulties:

Let $G$ be acyclic, and $1, \ldots, n$ an order of the vertices with only forward arcs. Add all backward arcs, that is $\hat{G} := G \cup B$, $B := \{ij,\ i > j\}$. Note that this order is coherent for $\hat{G}$, and a cycle with $\beta$ backward arcs is the disjoint union of vertex-sets of $\beta$ cycles each having 1 backward arc.

Consider now the polyhedron $Q(\hat{G}, -\infty, 1, c_{B,k}) \subseteq kQ(\hat{G}, -\infty, \infty, c_{B,1})$. It can have negative vertices! (We cannot avoid this, since since we want equality constraints for the dual problem.) By Theorem 2.2 $Q(\hat{G}, -\infty, \infty, c_{B,1})$ – which is now simply $\{x \in \mathbb{R}^n : x(P) \leq 1 \text{ for every path } P\}$ – has the integer decomposition property again, and by Theorem 2.1 $\mathcal{Q}(G, -\infty, 1, c_{B,k})$ is TDI. Now we can finish using Theorem 2.2 exactly like in the proof of Theorem 2.4. (Negative variables do not disturb, since by complementary slackness a primal optimal solution $x \in Q(\hat{G}, -\infty, 1, c_{B,k})$ satisfies $x(P) = k$ for all paths $P$ of an optimal path partition; because of $x \leq 1$, $x$ has at least $k$ different 1 entries; because of Theorem 2.2 we have $x = x^1 + \ldots + x^k$, $x^i \in Q(\hat{G}, -\infty, \infty, c_{B,1})$, and then the positive coordinates of the $x_i$ meet every path, and in different vertices for $i \neq j = 1, \ldots, n$.)

## 3.3 Path partitions and cycle covers

Berger and Hartman studied the two next-to-extreme cases of Berge's conjecture [5], [6]:"$k = 2$" and "$k = \lambda - 1$" – the $k = 1$ and $k = \lambda$ cases being completely settled, see the subsections 1.2, 1.3, 1.4, 1.5. It is somewhat discouraging for the continuation that the path partition and cycle cover versions in these extreme cases are completely unrelated. The following theorem allows a larger gap between $k$ and the longest path for strongly connected graphs, and shows some connection between path partitions and cycle covers. This is a result of an ongoing research with Irith Hartman in the frame of the French-Israeli collaboration project.

**Theorem 3.2** *Conjecture 3 is true provided $G$ is strongly connected and $k \geq \lambda - \sqrt{\lambda}$.*

**Proof**: Let $G$ be strongly connected, $k \geq \lambda - \sqrt{\lambda}$, and choose a coherent order. Apply Theorem 2.4 and let $S_1, \ldots, S_k$ the stable sets in the maximum, $X$ and $\mathcal{C}$ the set and cycle cover in the minimum, moreover, suppose that among the possible choices, $|X|$ is biggest possible. The problem is that the cycles in $\mathcal{C}$ are not necessarily disjoint.

If a cycle has at most $\lambda - \lfloor\sqrt{\lambda}\rfloor$ vertices not covered by any other cycle, then delete it from $\mathcal{C}$ and add to $X$ the vertices that get now uncovered, contradicting the choice of $X$. So the difference of any two cycles has size larger than $\lambda - \lfloor\sqrt{\lambda}\rfloor$.

*If for two intersecting cycles $C_1$, $C_2$ we have $|C_1 \setminus C_2|, |C_2 \setminus C_1| > \lambda - \lfloor\sqrt{\lambda}\rfloor$, then their union contains a path with more than $\lambda$ vertices.* (This bound is essentially tight.)

Indeed, we have then $|C_1 \cap C_2| \leq \lfloor\sqrt{\lambda}\rfloor$. But then $C_1 \cap C_2$ divides both cycles into paths, and one of these paths has at least $\lfloor\sqrt{\lambda}\rfloor$ vertices outside $C_1$, say. (If all these subpaths of $C_2$ have at most $\sqrt{\lambda} - 1$ vertices outside $C_1$, then $|C_2 \setminus C_1| \leq \sqrt{\lambda}(\sqrt{\lambda} - 1) = \lambda - \sqrt{\lambda}$.) Take such a path $P$ for instance in $C_2$. Then $|C_1 \cup P| > \lambda - \lfloor\sqrt{\lambda}\rfloor + \lfloor\sqrt{\lambda}\rfloor = \lambda$. It is easy to see that $|C_1 \cup P|$ contains a Hamiltonian path, that is, a path of length larger than $\lambda$, contradicting the definition of $\lambda$.

So the cycles in $\mathcal{C}$ are pairwise disjoint and then we are done again by complementary slackness.
□

Note that Theorem 1.12 cannot entirely replace Theorem 2.4 in this proof, because the version we stated (and we cannot prove more) does not suppose that $|X| + k\mathcal{C}$ is minimum, whereas $|X| + k\,\mathrm{ind}(\mathcal{C})$ is minimized in Theorem 2.4 and this fact is exploited in the proof.

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