Les cahiers Leibniz

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ISSN : 1298-020X

n° 175

décembre 2008

Site internet : http://www.g-scop.inpg.fr/CahiersLeibniz/

Asteroids in rooted and directed path graphs*

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December 15, 2008

Abstract

An asteroidal triple is a stable set of three vertices such that each pair is connected by a path avoiding the neighborhood of the third vertex. Asteroidal triples play a central role in a classical characterization of interval graphs by Lekkerkerker and Boland. Their result says that a chordal graph is an interval graph if and only if it contains no asteroidal triple. In this paper, we prove an analogous theorem for directed path graphs which are the intersection graphs of directed paths in a directed tree. For this purpose, we introduce the notion of a strong path. Two non-adjacent vertices are linked by a strong path if either they have a common neighbor or they are the endpoints of two vertex-disjoint chordless paths satisfying certain conditions. A strong asteroidal triple is an asteroidal triple such that each pair is linked by a strong path. We prove that a chordal graph is a directed path graph if and only if it contains no strong asteroidal triple. We also introduce a related notion of asteroidal quadruple, and conjecture a characterization of rooted path graphs which are the intersection graphs of directed paths in a rooted tree.

1 Introduction

A *hole* is a chordless cycle of length at least four. A graph is a *chordal graph* if it contains no hole as an induced subgraph. Gavril [4] proved that a graph is chordal if and only if it is the intersection graph of a family of subtrees of a tree. In this paper, whenever we talk about the intersection of subgraphs of a graph we mean that the *vertex sets* of the subgraphs intersect.

A graph is an *interval graph* if it is the intersection graph of a family of intervals on the real line; or equivalently, the intersection graph of a family of subpaths of a path. An *asteroidal triple* in a graph G is a set of three non-adjacent vertices such that for any two

*Research supported by NSERC

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of them, there exists a path between them in G that does not intersect the neighborhood of the third. The graph of Figure 1 is an example of a graph that minimally contains an asteroidal triple; the three vertices forming the asteroidal triple are circled.

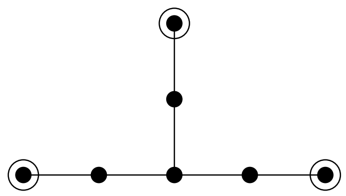


Figure 1: Graph containing an asteroidal triple

The following classical theorem was proved by Lekkerkerker and Boland.

Theorem 1 ([9]) *A chordal graph is an interval graph if and only if it contains no asteroidal triple.*

Lekkerkerker and Boland [9] derived from Theorem 1 the list of minimal forbidden subgraphs for interval graphs (see Figure 13).

The class of path graphs lies between interval graphs and chordal graphs. A graph is a *path graph* if it is the intersection graph of a family of subpaths of a tree. Lévêque, Maffray and Preissman [10] found a characterization of path graphs by forbidden subgraphs (see Figure 11).

Two variants of path graphs have been defined when the tree is a directed graph. A *directed tree* is a directed graph whose underlying undirected graph is a tree. A graph is a *directed path graph* if it is the intersection graph of a family of directed subpaths of a directed tree. Panda [14] found a characterization of directed path graphs by forbidden subgraphs (see Figure 12). A *rooted tree* is a directed tree in which the path from a particular vertex r to every other vertex is a directed path; vertex r is called the *root*. A graph is a *rooted path graph* if it is the intersection graph of a family of directed subpaths of a rooted tree. The problem of finding a characterization of rooted path graphs by forbidden subgraphs is still open.

Clearly, we have the following inclusions between the classes considered :

$$\text{interval} \subset \text{rooted path} \subset \text{directed path} \subset \text{path} \subset \text{chordal}$$

In this paper, we study directed path graphs and rooted path graphs. Our main result is a characterization of directed path graphs analogous to the theorem of Lekkerkerker and Boland. For this purpose, we introduce the notion of a strong path. Two non-adjacent vertices u and v are linked by a strong path if either they have a common neighbor or they are the endpoints of two vertex-disjoint chordless paths satisfying certain technical conditions. (The complete definition is given in Section 4.) A *strong asteroidal triple* in a graph G is an asteroidal triple such that each pair of vertices of the triple is linked by a strong path in G.

Our main result is the following theorem.

Theorem 2 *A chordal graph is a directed path graph if and only if it contains no strong asteroidal triple.*

In Section 2, we give the definitions and background results needed to prove our theorems. In Section 3, we give a new proof of Theorem 1 based on clique trees. In Section 4, we define strong paths and establish a property of strong paths in clique directed path trees (which are defined in Section 2). In Section 5, we give a proof of Theorem 2 using the results of Section 4. In sections 6 and 7, we discuss asteroidal quadruples and their relationship with graphs which are minimally not rooted path graph. Finally, in Section 8, we discuss new problems arising from our work.

2 Definitions and background

In a graph G, a *clique* is a set of pairwise adjacent vertices. Let $\mathcal{Q}(G)$ be the set of all (inclusionwise) maximal cliques of G. When there is no ambiguity we will write $\mathcal{Q}$ instead of $\mathcal{Q}(G)$. If a vertex u is adjacent to a vertex v, we say that u *sees* v; otherwise, we say u *misses* v. A vertex in a graph G is called *universal* if it sees every other vertex of G. Given a vertex v and a set S of vertices, v is called *complete to* S if v sees every vertex of S. Given two vertices u and v in a graph G, a $\{u, v\}$*-separator* is a set S of vertices of G such that u and v lie in two different components of $G \setminus S$ and S is minimal with this property. A set is a *separator* if it is a $\{u, v\}$-separator for some u and v in G. Let $\mathcal{S}(G)$ be the set of separators of G. When there is no ambiguity we will write $\mathcal{S}$ instead of $\mathcal{S}(G)$. A classical result [7, 1] (see also [6]) states that, in a chordal graph G, every separator is a clique; moreover, if S is a separator, then there are at least two components of $G \setminus S$ that contain a vertex that is complete to S, and so S is the intersection of two maximal cliques.

A *clique tree* T of a graph G is a tree whose vertices are the members of $\mathcal{Q}$ and such that, for each vertex v of G, those members of $\mathcal{Q}$ that contain v induce a subtree of T, which we will denote by T^v. A classical result [4] states that a graph is chordal if and only if it has a clique tree. A *clique path tree* T of G is a clique tree of G such that, for each vertex v of G, T^v is a path. Gavril [5] proved that a graph is a path graph if and only if it has a clique path tree. A *clique directed path tree* T of G is a clique path tree of G such that edges of the tree T are directed and for each vertex v of G, the subpath T^v is a directed path. A *clique rooted path tree* T of G is a clique directed path tree of G such that T is a rooted tree. Monma and Wei [13] proved that a graph is a directed path graph if and only if it has a clique directed path tree, and that a graph is a rooted path graph if and only if it has a clique rooted path tree. A *clique path* T of G is a clique tree of G such that T is a path. A graph is an interval graph if and only if it has a clique path [3]. These results allow us to consider only the intersection models that are clique trees when studying the properties of the graph classes.

For a clique tree T, the *label* of an edge QQ' of T is defined as $S_{QQ'} = Q \cap Q'$. Note that every edge QQ' satisfies $S_{QQ'} \in \mathcal{S}$; indeed, there exist vertices $v \in Q \setminus Q'$ and $v' \in Q' \setminus Q$ such that the set $S_{QQ'}$ is a $\{v, v'\}$-separator. The number of times an element S of $\mathcal{S}$ appears as a label of an edge is equal to $c - 1$, where c is the number of components of $G \setminus S$ that contain a vertex complete to S [4, 12]. Note that this number is at least one and that it depends only on S and not on T, so for a given $S \in \mathcal{S}$ it is the same in every clique tree. For more information about clique trees and chordal graphs, see [6, 12].

Given $X \subseteq \mathcal{Q}$, let $G(X)$ denote the subgraph of G induced by all the vertices that appear in the members of X. If T is a clique tree of G, then $T[X]$ denotes the subtree of T of minimum size whose vertices contains X. Note that if $|X| = 2$, then $T[X]$ is a path. Given a subtree T' of a clique tree T of G, let $\mathcal{Q}(T')$ be the set of vertices of T' and $\mathcal{S}(T')$ be the set of separators of $G(\mathcal{Q}(T'))$. Note that T' is a clique tree of $G(\mathcal{Q}(T'))$.

Given a set $z_1, \dots, z_r$, $r \geq 2$, of pairwise non-adjacent vertices of G, and a clique tree T of G, the subtrees T^{z_i}, $1 \leq i \leq r$, are disjoint and we can define $T(z_1, \dots, z_r)$ the subtree of T of minimum size that contains at least one vertex of each T^{z_i}. Clearly, the number of leaves of $T(z_1, \dots, z_r)$ is at most r. Moreover, if $T(z_1, \dots, z_r)$ has exactly r leaves, then they can be denoted by Q_i, $1 \leq i \leq r$, with $Q_i \cap \{z_1, \dots, z_r\} = \{z_i\}$.

3 Asteroidal triples

In this section, we give a proof of Theorem 1 using clique trees. First, we need the following lemma which is folklore (for example, see [11].)

Lemma 1 *Let G be a chordal graph and z_1, z_2, z_3 three vertices that form an asteroidal triple, then for every clique tree T of G, the subtree $T(z_1, z_2, z_3)$ has exactly 3 leaves.*

Proof. Suppose the subtree $T(z_1, z_2, z_3)$ is a path. For $1 \leq i \leq 3$, let Q_i be a vertex of $T(z_1, z_2, z_3)$ containing z_i (they are all distinct as they are cliques of G and z_1, z_2, z_3 are not adjacent). We can assume that Q_1, Q_2, Q_3 appear in this order along the path $T(z_1, z_2, z_3)$. Vertices z_1 and z_3 are in two different components of the graph $G \setminus Q_2$ so every path that goes from z_1 to z_3 has to use a vertex of Q_2 and thus a neighbor of z_2, contradicting the fact that z_1, z_2, z_3 is an asteroidal triple. $\square$

A consequence of Lemma 1 is that an interval graph does not contain an asteroidal triple. Lekkerkerker and Boland [9] proved that the converse is also true. Halin [8] gave a short proof of Theorem 1. Unfortunately, this proof is hard to follow as it uses the so called *prime graph decomposition*. In the book [12], McKee and McMorris shorten the proof of Halin by considering the clique tree, but this proof is incomplete. A correction proposed by the authors afterward on the book's website is also incomplete.

It seems to be easier to understand Halin's proof in terms of clique trees rather than in terms of his prime graph decomposition. We are going to give such a proof. The ideas of this proof have been generalized and extensively used in [10] to obtain a forbidden subgraph characterization of path graphs.

Proof of Theorem 1. *($\Longrightarrow$)* Suppose G is an interval graph and z_1, z_2, z_3 is an asteroidal triple of G. Let T be a clique path of G. By Lemma 1, the subtree $T(z_1, z_2, z_3)$ has exactly 3 leaves, so T is not a path, a contradiction.

($\Longleftarrow$) Suppose that G is a chordal graph containing no asteroidal triple and that G is a minimal non-interval graph. Let T be any clique tree of G. The graph G is not an interval graph, so T has at least three distinct and non-adjacent leaves Q_1, Q_2, Q_3. For $i = 1, 2, 3$, let Q'_i be the neighbor of Q_i on T. (The vertices Q'_i are distinct from Q_1, Q_2, Q_3 but not necessarily pairwise distinct.) For $i = 1, 2, 3$, let $z_i \in Q_i \backslash Q'_i$ and $S_i = Q_i \cap Q'_i$. Vertices z_1, z_2, z_3 are three non-adjacent vertices that do not form an asteroidal triple. So, by symmetry, we can assume that every path that goes from z_2 to z_3 use some vertices in $N(z_1)$.

We claim that there is an edge of $T[Q_2, Q_3]$ such that z_1 is complete to the label of the edge. (Recall that a label is a set of vertices. So here, z_1 is complete to the set of vertices which are the label of the edge.) Suppose, on the contrary, that there is no edge of $T[Q_2, Q_3]$ such that z_1 is complete to its label. Then, in each label of edges of $T[Q_2, Q_3]$, one can select a vertex of G that is not adjacent to z_1. The set of selected vertices forms a path from z_2 to z_3 that uses no vertex from $N(z_1)$, a contradiction. So z_1 is complete to the label S_0 of an edge of $T[Q_2, Q_3]$, so $S_0 \subseteq S_1$. The vertex Q_1 is not on $T[Q_2, Q_3]$, so S_0 and S_1 are labels of two different edges of T, even if they may be equal.

Let P be a clique path of $G(\mathcal{Q}(T) \backslash \{Q_1\})$. Let T' be the clique tree of G obtained from P by adding vertex Q_1 and edge $Q_1 Q'_1$. For the two clique trees T and T' of G, the number of times a label appears in each clique tree is the same, so S_0, S_1 are labels of two different edges of T' and so S_0 is the label of an edge of P.

Let P' be the maximal subpath of P that contains Q'_1 and such that no label of edges of P' is a subset of S_1. Let T_0 be the clique tree of $G(\mathcal{Q}(P') \cup \{Q_1\})$ obtained from P' by adding vertex Q_1 and edge $Q_1 Q'_1$. As Q_1 is a leaf of T_0, every label of T_0 that is a subset of Q_1 is a subset of S_1. Since only one label of T_0 is a subset of S_1, only one label of T_0 is a subset of Q_1. The label S_0 is a subset of S_1, and so P' has strictly fewer vertices than P. So $G(\mathcal{Q}(P') \cup \{Q_1\})$ is an interval graph. Let P_0 be a clique path of this graph. For the two clique trees T_0, P_0 of $G(\mathcal{Q}(P') \cup \{Q_1\})$, the number of times a label appears in each clique tree is the same, so only one label of P_0 is a subset of Q_1. So Q_1 is a leaf of P_0.

The path P' is a proper subpath of P, so $P \backslash P'$ is either a path or the union of two paths.

Case 1 : $P \backslash P'$ is a path. Let P_1 be the path $P \backslash P'$. Let L be the leaf of P_1 such that there exists a vertex L' in P' with LL' being an edge of P. By definition of P', the label $S_{LL'}$ of LL' is included in S_1. So P_0 and P_1 can be linked by the edge LQ_1 to obtain a clique path of G, a contradiction.

Case 2 : $P\backslash P'$ is the union of two paths. Let P_1, P_2 the paths of $P\backslash P'$. Let L_i be the leaf of P_i such that there exists a vertex L'_i in P' with $L_iL'_i$ being an edge of P. By definition of P', labels $S_{L_iL'_i}$ of $L_iL'_i$ are subsets of S_1. Let L_0 be the leaf of P_0 that is different from Q_1. The vertex L_0 is a vertex of P', so it is either on $P'[Q'_1, L'_1]$ or on $P'[Q'_1, L'_2]$. Suppose, by symmetry, that L_0 is on $P'[Q'_1, L'_2]$. Since every vertex of the path $P'[Q'_1, L'_2]$ contains $S_{L_2L'_2}$, vertex L_0 contains $S_{L_2L'_2}$. So P_0, P_1 and P_2 can be linked by edges L_1Q_1, L_2L_0 to obtain a clique path of G, a contradiction. □

4 Strong paths and clique directed path trees

Two non-adjacent vertices u and v are linked by a *strong path* if either they have a common neighbor, or there exist three sets of distinct vertices $X = \{x_1, \dots, x_r\}, Y = \{y_1, \dots, y_s\}, Z$, $(r, s \geq 2,\ |Z| \geq 0)$, such that u-x_1-$\cdots$-x_r-v and u-y_1-$\cdots$-y_s-v are two chordless paths and for every clique of size four $x_i, x_{i+1}, y_j, y_{j+1}$ $(1 \leq i < r,\ 1 \leq j < s)$, one of the following is satisfied with $\{l_1, l_2\} = \{x_i, y_j\}$ and $\{r_1, r_2\} = \{x_{i+1}, y_{j+1}\}$:

- *Attachment of type 1 on $\{l_1, l_2\}, \{r_1, r_2\}$:* There exist two non-adjacent vertices z, z' of Z such that vertex z sees l_1, l_2, r_1 and not r_2, vertex z' sees r_1, r_2, l_1 and not l_2.
- *Attachment of type 2 on $\{l_1, l_2\}, \{r_1, r_2\}$:* There exist $4t + 3$ $(t \geq 0)$ vertices $z_1, \dots, z_{2t+1}, z'_1, \dots, z'_{2t+2}$ of Z such that vertices $l_1, l_2, r_1, r_2, z_1, \dots, z_{2k+1}$ form a clique Q, vertices $z_1, \dots, z_{2t+1}$ sees exactly l_1, l_2, r_1, r_2 on $X \cup Y \cup \{u, v\}$, vertices $z'_1, \dots, z'_{2t+2}$ form a stable set, vertex z'_k $(1 \leq k \leq 2t + 2)$ sees exactly z_{k-1}, z_k on $Q \cup X \cup Y \cup \{u, v\}$ (with $z_0 = l_1$ and $z_{2t+2} = r_1$).

The graphs of Figure 2 are examples of chordal graphs in which vertices u and v are linked by a strong path.

Strong paths are interesting when considering directed path graphs because of the following lemma.

Lemma 2 *Let G be a directed path graph and u and v two non-adjacent vertices that are linked by a strong path, then for every clique directed path tree T of G, the subpath $T(u, v)$ is a directed path.*

Proof. Suppose on the contrary that T is a clique directed tree of G such that $T(u, v)$ is not a directed path. Let Q_u and Q_v be the two extremities of $T(u, v)$ with $u \in Q_u$ and $v \in Q_v$. By assumption, the subpath $T(u, v)$ contains two edges not oriented in the same direction. Let Q_1, Q_2, Q_3 be three consecutive vertices of $T(u, v)$ such that Q_u, Q_1, Q_2, Q_3, Q_v appears in this order along $T(u, v)$ (we may have $Q_u = Q_1$, $Q_3 = Q_v$) and edges Q_1Q_2 and Q_2Q_3 are not oriented in the same direction. By reversing all the

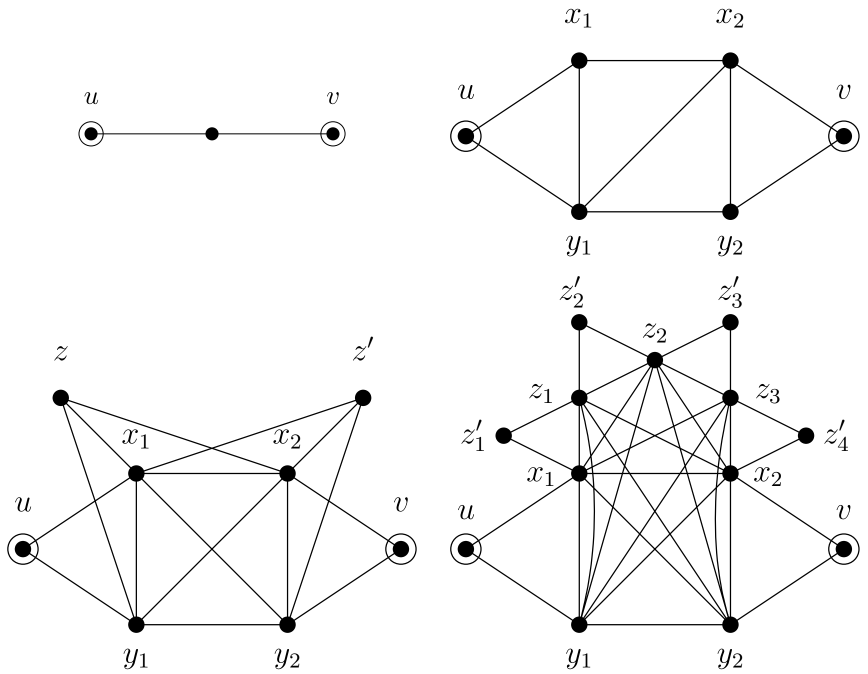


Figure 2: Examples of strong paths

edges of T if necessary, we may assume that $Q_1 \to Q_2$ and $Q_2 \leftarrow Q_3$ (where $Q_1 \to Q_2$ mean there is a edge directed from Q_1 to Q_2). As T is a clique directed path tree, $S_{Q_1Q_2} \cap S_{Q_2Q_3} = \emptyset$.

Suppose that there exists $w \in N(u) \cap N(v)$. Then w is not in $S_{Q_1Q_2}$ or not in $S_{Q_2Q_3}$. By symmetry, we may assume that $w \notin S_{Q_1Q_2}$. Then u and v are in two components of $G \backslash S_{Q_1Q_2}$, a contradiction because u-w-v is a path. So, we have $N(u) \cap N(v) = \emptyset$ and there exists a strong path with $X = \{x_1, \ldots, x_r\}, Y = \{y_1, \ldots, y_s\}, Z$, $(r, s \geq 2, |Z| \geq 0)$.

We claim that every label of the edges of $T(u, v)$ contains at least one vertex from X and at least one vertex from Y. Suppose on the contrary that there exists an edge QQ' of $T(u, v)$ such that $S_{QQ'} \cap X = \emptyset$. Then u and v are in two different components of $G \backslash S_{QQ'}$, contradicting the fact that u-X-v is a path. The case for Y is similar and thus our claim holds.

Let i and j be the maximum subscripts with $1 \leq i \leq r$, $1 \leq j \leq s$ such that $x_i, y_j \in S_{Q_1Q_2}$. As $S_{Q_1Q_2} \cap S_{Q_2Q_3} = \emptyset$, we have $x_i, y_j \in S_{Q_1Q_2} \backslash S_{Q_2Q_3}$. Vertex v is not in Q_2 by definition of $T(u, v)$, so not in $S_{Q_2Q_3}$. In $G \backslash S_{Q_2Q_3}$, vertices x_i, y_j are not in the same component as v, so x_i, y_j are not adjacent to v and $i < r$, $j < s$.

The set $S_{Q_2Q_3}$ contains at least one vertex from X and at least one vertex from Y. Vertices of $S_{Q_2Q_3}$ are in Q_2, and thus are adjacent to both x_i, y_j. As X and Y are chordless paths, $S_{Q_2Q_3}$ contains at least one of x_{i-1}, x_{i+1} and at least one of y_{j-1}, y_{j+1}.

Suppose that $S_{Q_2Q_3}$ contains x_{i-1} (so $i > 2$). Then x_{i-1} is not in $S_{Q_1Q_2}$. Because $S_{Q_1Q_2}$ contains x_i and u-x_1-$\cdots$-x_{i-1}-x_i is a chordless path, $S_{Q_1Q_2}$ contains no vertex of $\{u, x_1, \dots, x_{i-2}\}$. Then u and x_{i-1} are in two different components of $G \backslash S_{Q_1Q_2}$, contradicting the fact that u-x_1-$\cdots$-x_{i-1} is a path. So, we have $x_{i-1} \notin S_{Q_2Q_3}$, and $x_{i+1} \in S_{Q_2Q_3}$. Similarly, we have $y_{j+1} \in S_{Q_2Q_3}$.

Now, the vertices $x_i, y_j, x_{i+1}, y_{j+1}$ form a clique of size four in $X \cup Y$. So, there are vertices of Z forming an attachment of type 1 or 2 on $\{x_i, y_j\}, \{x_{i+1}, y_{j+1}\}$.

Suppose first that there is an attachment of type 1. Then, by symmetry, we may assume that there exist two non-adjacent vertices z, z' of Z such that z sees x_i, y_j, x_{i+1} and not y_{j+1} and z' sees x_i, x_{i+1}, y_{j+1} and not y_j. Let Q_z be a vertex of T containing z, x_i, y_j, x_{i+1} and $Q_{z'}$ a vertex of T containing $z', x_i, x_{i+1}, y_{j+1}$. All of $Q_1, Q_2, Q_z, Q_{z'}$ contain x_i so there are all vertices of the path T^{x_i}. Vertex Q_1 is not between Q_z and Q_2 or between $Q_{z'}$ and Q_2 along this path, since otherwise $S_{Q_1Q_2}$ contains x_{i+1}. Vertex Q_z is not between Q'_z and Q_2, since otherwise z sees y_{j+1}. So vertices $Q_1, Q_2, Q_{z'}, Q_z$ appear in this order along the path T^{x_i}, but then z' sees y_j, a contradiction.

Suppose now that there is an attachment of type 2. Then, by symmetry, we can assume that there exist $4t+3$ ($t \geq 0$) vertices $z_1, \dots, z_{2t+1}, z'_1, \dots, z'_{2t+2}$ of Z such that vertices $x_i, y_j, x_{i+1}, y_{j+1}, z_1, \dots, z_{2t+1}$ form a clique Q, vertices $z_1, \dots, z_{2t+1}$ see exactly $x_i, y_j, x_{i+1}, y_{j+1}$ on $X \cup Y \cup \{u, v\}$, vertices $z'_1, \dots, z'_{2t+2}$ form a stable set, vertex z'_k ($1 \leq k \leq 2t+2$) sees exactly z_{k-1} and z_k on $Q \cup X \cup Y \cup \{u, v\}$ (with $z_0 = x_i$ and $z_{2t+2} = x_{i+1}$). Let K be a vertex of T containing Q (we may have $K = Q_2$) and for $1 \leq k \leq 2t+2$, let Z_k be a vertex of T containing z'_k, z_{k-1}, z_k.

Vertices Q_1, Q_2, K are all on the path T^{x_i}. If Q_1 is between K and Q_2, then $S_{Q_1Q_2}$ contains y_{j+1}, a contradiction. So Q_1, Q_2, K appear in this order along T^{x_i}. Vertex Z_1 is also on T^{x_i}. Vertex Z_1 is not between Q_1 and K as it does not contain y_j.

Suppose that Z_1, Q_1, Q_2, K appear in this order along T^{x_i}. If Z_1 is on T^{y_j}, then z'_1 sees y_j, a contradiction. So Z_1 is not on T^{y_j}. Let Q'_1 be the clique of $T(Z_1, Q_1)$ nearest to Z_1 that contains y_j (we may have $Q'_1 = Q_1$). If $Q'_1 = Q_u$, then z_1 sees u, a contradiction. So $Q'_1 \neq Q_u$. Let Z'_1 be the neighbor of Q'_1 on $T(Z_1, Q'_1)$. If Z'_1 is a vertex of $T(u, v)$, then $S_{Z'_1Q'_1}$ contains a vertex of Y that sees y_j and that is different from y_{j+1}, so $S_{Z'_1Q'_1}$ contains y_{j-1} and z_1 sees y_{j-1}, a contradiction. So Z'_1 is not a vertex of $T(u, v)$. Let Q''_1 be the clique of $T(Q'_1, Q_1)$ nearest to Q'_1 that is in $T(u, v)$ (we may have $Q''_1 = Q_1$ or $Q''_1 = Q'_1$ or both). If $Q''_1 = Q_u$, then z_1 sees u, a contradiction. Let Q'_u be the neighbor of Q''_1 on $T(u, v)$ that is not on $T(Z_1, Q_2)$. Then $S_{Q'_uQ''_1}$ contains a vertex of X that sees x_i and that is different from x_{i+1}, so $S_{Q'_uQ''_1}$ contains x_{i-1} and z_1 sees x_{i-1}, a contradiction. So Q_1, Q_2, K, Z_1 appear in this order along T^{x_i}. Similarly, Q_3, Q_2, K, Z_{2t+2} appear in this order along $T^{x_{i+1}}$.

For every k and ℓ, $1 \leq k < \ell \leq 2t+2$, vertices Z_k, K, Z_ℓ appear in this order along $T[K, Z_k, Z_\ell]$, since otherwise, Z_k contains z_ℓ or Z_ℓ contains z_{k-1}. Vertices Q_1, K, Z_1 appear in this order along T^{x_i} and $Q_1 \to Q_2$, so $T[K, Z_1]$ is directed from K to Z_1. Vertices Z_1, K, Z_2 appear in this order along T^{z_1}, so $T[K, Z_2]$ is directed from Z_2 to K. And so on, for $2 \leq k \leq 2t+2$, vertices Z_{k-1}, K, Z_k appear in this order along T^{z_k}, so

$T[K, Z_k]$ is directed from K to Z_k when k is odd and from Z_k to K when k is even. So $T[K, Z_{2t+2}]$ is directed from Z_{2t+2} to K. Vertices Q_3, Q_2, K, Z_{2t+2} appear in this order along $T^{x_{i+1}}$, so $T[K, Q_3]$ is directed from K to Q_3, contradicting $Q_2 \leftarrow Q_3$. $\square$

5 Asteroidal triples in directed path graphs

The graph of Figure 1 is a directed path graph that is minimally not an interval graph; in fact, it is a rooted path graph. (It is the graph F_{18} of Figure 13.) So, directed path graphs may contain asteroidal triples. But one can define a particular type of asteroidal triples that is forbidden in directed path graphs. Recall from Section 1 that a *strong asteroidal triple* in a graph G is an asteroidal triple such that each pair of vertices of the triple are linked by a strong path in G. The graph of Figure 3 is example of a graph that minimally contains a strong asteroidal triple. This graph is a path graph which is minimally not a directed path graph. (It is the graph $F_{17}(6)$ of Figure 12, and also $F_{21}(6)$ of Figure 13.)

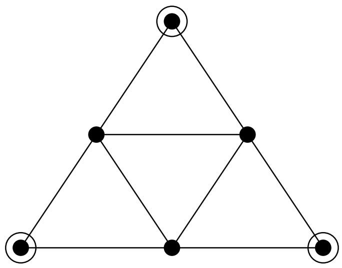


Figure 3: Graph containing a strong asteroidal triple

The graph of Figure 4 is another example of a graph that minimally contains a strong asteroidal triple. This graph is interesting as it shows that sometimes the path between two vertices of the asteroidal triple that avoids the neighborhood of the third must contain some vertices outside the strong path. The only strong path linking 2 and 3 is $X = \{x_1, x_2\}, Y = \{y_1, y_2\}, Z = \emptyset$ and the only path between 2 and 3 that avoids the neighborhood of 1 is y_1-t-x_2. This graph is a chordal graph which is minimally not a path graph. (It is the graph $F_{10}(8)$ of Figures 11, 12 and $F_{21}(8)$ of Figure 13.)

The graph of Figure 1 is an example of a graph that contains an asteroidal triple that is not strong as for two vertices of the asteroidal triple, there is no common neighbor and no pair of disjoint paths between them. The graph of Figure 5 is another example of a graph that contains an asteroidal triple that is not strong. In this graph, there exist two disjoint paths $\{x_1, x_2\}, \{y_1, y_2\}$ between 2 and 3 but x_1, x_2, y_1, y_2 is a clique of size four and there are no vertices that can play the role of Z in the definition of strong path. This graph is a rooted path graph which is minimally not an interval graph. (It is the graph $F_{21}(7)$ of Figure 13.)

We will now prove Theorem 2 which gives a characterization of directed path graphs

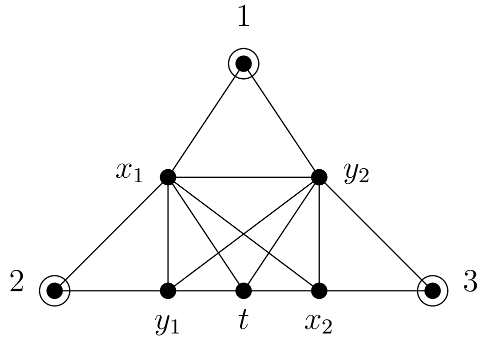


Figure 4: Graph containing a strong asteroidal triple

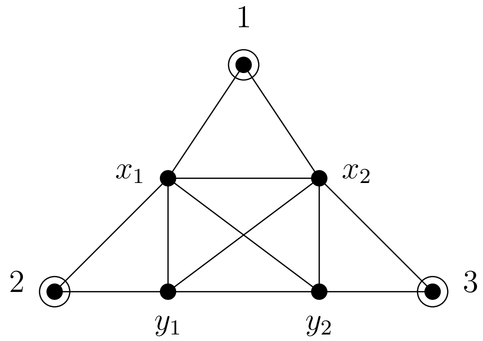


Figure 5: Graph containing an asteroidal triple that is not strong

using strong asteroidal triples.

Proof of Theorem 2. ($\Longrightarrow$) Suppose that G is a directed path graph and z_1, z_2, z_3 is a strong asteroidal triple of G. Let T be a clique directed path tree of G. By Lemma 1, $T(z_1, z_2, z_3)$ has exactly 3 leaves Q_i, $1 \leq i \leq 3$, with $z_i \in Q_i$. By Lemma 2, $T(z_1, z_2)$, $T(z_2, z_3)$, $T(z_3, z_1)$ are directed paths of $T(z_1, z_2, z_3)$. Suppose, by symmetry, that $T(z_1, z_2)$ is directed from Q_1 to Q_2. Then $T(z_1, z_3)$ is directed from Q_1 to Q_3, but then $T(z_2, z_3)$ is not a directed path, a contradiction.

($\Longleftarrow$) All chordal graphs of Figure 12 contain a strong asteroidal triple. The graphs F_1, F_3, F_4, $F_5(n)_{n\geq 7}$ are obtained from a graph containing an asteroidal triple by adding a universal vertex; this universal vertex forms a strong path linking each pair of vertices of the asteroidal triple. In the graphs F_6, F_7, F_9, $F_{10}(n)_{n\geq 8}$, the strong paths are either a common neighbor or four vertices without attachment. In the graphs $F_{13}(4k+1)_{k\geq 2}$, $F_{15}(4k+2)_{k\geq 2}$, $F_{16}(4k+3)_{k\geq 2}$, $F_{17}(4k+2)_{k\geq 1}$, the strong paths are either a common neighbor, four vertices without attachment, or four vertices plus attachment of type 1 or 2. So if G is a chordal graph containing no strong asteroidal triple, it contains no F_1, F_3, F_4, $F_5(n)_{n\geq 7}$, F_6, F_7, F_9, $F_{10}(n)_{n\geq 8}$, $F_{13}(4k+1)_{k\geq 2}$, $F_{15}(4k+2)_{k\geq 2}$, $F_{16}(4k+3)_{k\geq 2}$, $F_{17}(4k+2)_{k\geq 1}$, and so it is a directed path graph by the result of Panda [14]. $\square$

In the proof of Theorem 2, we use the list of forbidden subgraph obtained by

Panda [14]. It would be nice to find a simple and direct proof of this result, similar to the proof of Theorem 1 presented in Section 3.

A corollary of Theorem 2 and [14] is the following.

Corollary 1 *The chordal graphs that minimally contain a strong asteroidal triple are the graphs* F_1, F_3, F_4, $F_5(n)_{n\geq 7}$, F_6, F_7, F_9, $F_{10}(n)_{n\geq 8}$, $F_{13}(4k+1)_{k\geq 2}$, $F_{15}(4k+2)_{k\geq 2}$, $F_{16}(4k+3)_{k\geq 2}$, $F_{17}(4k+2)_{k\geq 1}$.

One can notice that only *short strong paths* are used in the proof of Theorem 2 (where *short* means either a common neighbor or $|X| = |Y| = 2$). So if one is interested only in characterizing directed path graphs, there is no need to define *long strong path* (where *long* means one of $|X|$, $|Y|$ is at least 3). As we will see in next sections, long strong paths are useful for rooted path graphs.

6 Asteroidal quadruples in rooted path graphs

The notion of asteroidal triple can be generalized to four vertices. An *asteroidal quadruple* in a graph G is a set of four vertices such that any three of them is an asteroidal triple. The graph of Figure 6 is an example of a graph that minimally contains an asteroidal quadruple.

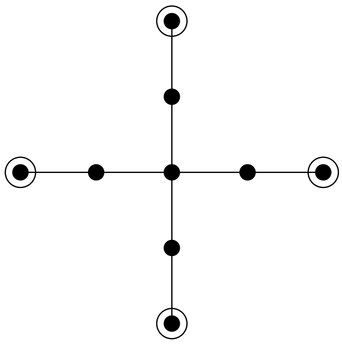


Figure 6: Graph containing an asteroidal quadruple

The following lemma is analogous to Lemma 1 for asteroidal quadruple (see also [11]).

Lemma 3 *Let G be a chordal graph and let z_1, z_2, z_3, z_4 be four vertices that form an asteroidal quadruple. Then for every clique tree T of G, the subtree $T(z_1, z_2, z_3, z_4)$ has exactly 4 leaves.*

Proof. Suppose that the subtree $T(z_1, z_2, z_3, z_4)$ has fewer than 4 leaves. Then there is at least one z_i that is not in a leaf. Suppose by symmetry that z_4 is not in a leave. Let Q_4 be a vertex of $T(z_1, z_2, z_3, z_4)$ that contains z_4. Then $T(z_1, z_2, z_3, z_4) = T(z_1, z_2, z_3)$ and by Lemma 1, $T(z_1, z_2, z_3)$ has exactly 3 leaves. Vertex Q_4 is either on $T(z_1, z_2)$ or $T(z_1, z_3)$. Suppose by symmetry that Q_4 is a vertex of $T(z_1, z_2)$. Then $T(z_1, z_2, z_4) = T(z_1, z_2)$ is a path, contradicting Lemma 1. $\square$

The graph of Figure 6 is a rooted path graph, so a rooted path graph may contain asteroidal quadruples. But one can define a particular type of asteroidal quadruple that is forbidden in rooted path graphs.

One can try to use the notion of strong asteroidal triple to define a *strong asteroidal quadruple* as a set of four vertices such that any three of them is an strong asteroidal triple. This is not interesting for our purpose because then every graph that contains a strong asteroidal quadruple also contains a strong asteroidal triple. And, by Theorem 2, we already know that directed path graphs and thus rooted path graphs contain no strong asteroidal triple.

One can define another four-vertex variant of asteroidal triple that will be useful. A *weak asteroidal triple* in a graph G is an asteroidal triple such that two vertices of the asteroidal triple are linked by a strong path in G. The difference from the definition of a strong asteroidal triple is that we do not expect that there is a strong path linking any two of the three vertices but just linking two of them.

Now we can generalize this notion to four vertices. A *weak asteroidal quadruple* is a set of four vertices such that any three of them is a weak asteroidal triple. Weak asteroidal quadruples are interesting when considering directed path graphs because of the following theorem.

Theorem 3 *A rooted path graph contains no weak asteroidal quadruple.*

Proof. Suppose that G is a rooted path graph and z_1, z_2, z_3, z_4 is a weak asteroidal quadruple of G. Let T be a clique rooted path tree of G. By Lemma 3, $T(z_1, z_2, z_3, z_4)$ has exactly 4 leaves Q_i, $1 \leq i \leq 4$, with $z_i \in Q_i$. A subtree of a rooted tree is a rooted tree, so $T(z_1, z_2, z_3, z_4)$ is also rooted.

By definition of weak asteroidal quadruple, the three vertices z_1, z_2, z_3 form a weak asteroidal triple, so two of them are linked by a strong path. Suppose, by symmetry, that there is a strong path linking z_1 and z_2. Then, by Lemma 2, $T(z_1, z_2)$ is a directed path. So, Q_1 or Q_2 is the root of $T(z_1, z_2, z_3, z_4)$. Suppose by symmetry that Q_1 is the root of $T(z_1, z_2, z_3, z_4)$.

The three vertices z_2, z_3, z_4 also form a weak asteroidal triple, so there is a strong path linking two of them. Let z_i and z_j be two vertices of z_2, z_3, z_4 that are linked by a strong path. Then, by Lemma 2, $T(z_i, z_j)$ is a directed path; so Q_i or Q_j is the root of $T(z_1, z_2, z_3, z_4)$, a contradiction. $\square$

By Theorems 2 and 3, we know that a rooted path graph contains no hole, no strong asteroidal triple and no weak asteroidal quadruple. We conjecture that the converse is also true.

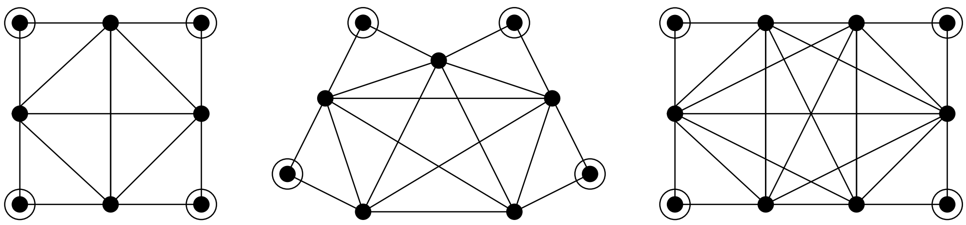
Conjecture 1 *A chordal graph is a rooted path graph if and only if it contains no strong asteroidal triple and no weak asteroidal quadruple.*

If Conjecture 1 is true, it will give a characterization of rooted path graphs analogous to our Theorem 2 on directed path graphs and to Lekkerkerker and Boland's characterization of interval graphs.

7 Forbidden subgraphs of rooted path graphs?

Recall that the problem of finding the forbidden induced subgraph characterization is solved for interval graphs, path graphs, and directed path graphs, but not for rooted path graphs. Whether Conjecture 1 is true or false, Theorems 2 and 3 can be used to obtain many graphs that are minimally not rooted path graphs. What are the graphs that contain a strong asteroidal triple or a weak asteroidal quadruple and that are minimally not rooted path graphs?

The graphs of Figure 7 are examples of graphs that minimally contain a weak asteroidal quadruple. They are also minimally not rooted path graphs. They are of particular interest since together with $F_{17}(6)$ (see Figure 3) they show that rooted path graphs contain no suns. A *sun* is the graph obtained by taking a clique on vertices $a_1, a_2, \ldots, a_k$ for some $k \geq 3$, a stable set on vertices $s_1, s_2, \ldots, s_k$, and adding edges $s_i a_i, s_i a_{i+1}$ for all i, with the subscripts taken modulo k. Farber showed [2] that a graph is strongly chordal if and only if it is chordal and does not contain a sun. Thus, rooted path graphs are a subclass of strongly chordal graphs.



F_{22} F_{23} F_{24}

Figure 7: Examples of graphs which are minimally not rooted path graphs and contain a weak asteroidal quadruple

Corollary 1 gives the list of chordal graphs that minimally contain a strong asteroidal triple. One may notice that $F_{13}(4k+1)_{k\geq 3}$, $F_{15}(4k+2)_{k\geq 2}$, $F_{16}(4k+3)_{k\geq 2}$, and $F_{17}(4k+2)_{k\geq 2}$ all strictly contain a weak asteroidal quadruple. All the other chordal graphs that minimally contain a strong asteroidal triple are minimally not rooted path graphs (see Figure 8).

Proposition 1 *The chordal graphs that are minimally not rooted path graphs and that contain a strong asteroidal triple are F_1, F_3, F_4, $F_5(n)_{n\geq 7}$, F_6, F_7, F_9, $F_{10}(n)_{n\geq 8}$, $F_{13}(9)$, $F_{17}(6)$.*

Proof. The graphs F_1, F_3, F_4, $F_5(n)_{n\geq 7}$, F_6, F_7, F_9, $F_{10}(n)_{n\geq 8}$, $F_{13}(9)$, and $F_{17}(6)$ contain a strong asteroidal triple, so they are forbidden in directed path graphs by Theorem 2, and thus forbidden in rooted path graphs. We omit the proof of minimality; one has to check that, for each of these graphs, one can remove any vertex and then one can find a clique rooted path tree for the resulting graph.

Suppose there is a chordal graph G that is minimally not a rooted path graph, that contains a strong asteroidal triple and that is different from F_1, F_3, F_4, $F_5(n)_{n\geq 7}$, F_6, F_7, F_9, $F_{10}(n)_{n\geq 8}$, $F_{13}(9)$, and $F_{17}(6)$. Since G is minimally not a rooted path graph, G does not strictly contain any of the graphs listed. By Corollary 1, G contains one of $F_{13}(4k+1)_{k\geq 3}$, $F_{15}(4k+2)_{k\geq 2}$, $F_{16}(4k+3)_{k\geq 2}$, or $F_{17}(4k+2)_{k\geq 2}$. But all these graphs strictly contain F_{23} of Figure 7. The graph F_{23} is not a rooted path graph, so G fails to be minimally not a rooted path graph, a contradiction. □

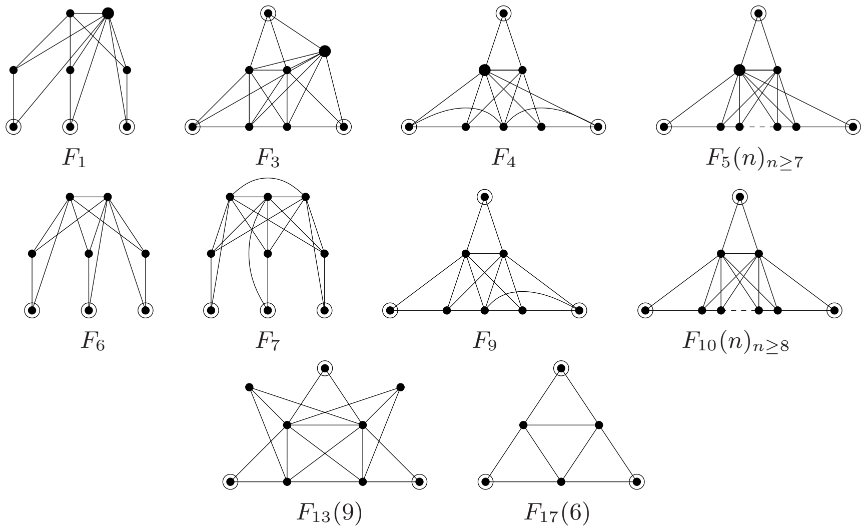


Figure 8: Minimal forbidden induced subgraphs of rooted path graphs that are chordal and contain a strong asteroidal triple

Now that we know the complete list of minimal forbidden induced subgraphs of rooted path graphs that are chordal and contain a strong asteroidal triple, we can look at those that contain a weak asteroidal quadruple. In this case, we have not obtained a complete list. We need the following lemma for which we omit the proof.

Lemma 4 *If G is a graph on four vertices such that any of its induced subgraphs on three vertices contains at least one edge, then G contains a triangle or two disjoint edges.*

A *parallel asteroidal quadruple* is an asteroidal quadruple z_1, z_2, z_3, z_4 such that there is a strong path linking z_1 and z_2 and a strong path linking z_3 and z_4. Parallel asteroidal quadruples are a particular type of weak asteroidal quadruple. By replacing edges by strong paths in Lemma 4, it is easy to see that a weak asteroidal quadruple either contains a strong asteroidal triple or is a parallel asteroidal quadruple. So we have the following corollary.

Corollary 2 *If a chordal graph does not contain a strong asteroidal triple but does contain a weak asteroidal quadruple, then it contains a parallel asteroidal quadruple.*

The following lemma shows that we do not need to consider attachments of type 2 when dealing with graphs that are minimally not rooted path graphs and contain no strong asteroidal triple.

Lemma 5 *Let G be a chordal graph that is minimally not a rooted path graph and that contains no strong asteroidal triple. If u and v are linked by a strong path, then they are linked by a strong path that uses no attachment of type 2.*

Proof. First, notice that the graph F_{23} satisfies the hypothesis of the lemma, so we can assume that G is different from F_{23}. Also G does not strictly contain F_{23} as G is minimally not a rooted path graph. Suppose u and v are linked by a strong path. We may assume that u and v have no common neighbors, for otherwise the lemma is true. Let X, Y, Z be a strong path linking u and v such that $|X \cup Y|$ is minimum. We use the same notation as in the definition of strong path. Suppose there is a clique of size four $x_i, y_j, x_{i+1}, y_{j+1}$ of $X \cup Y$ with an attachment of type 2. We may assume that there exist $4t + 3$ ($t \geq 0$) vertices $z_1, \ldots, z_{2t+1}, z'_1, \ldots, z'_{2t+2}$ of Z such that vertices $x_i, y_j, x_{i+1}, y_{j+1}, z_1, \ldots, z_{2t+1}$ form a clique Q, vertices $z_1, \ldots, z_{2t+1}$ see exactly $x_i, y_j, x_{i+1}, y_{j+1}$ on $X \cup Y \cup \{u, v\}$, vertices $z'_1, \ldots, z'_{2t+2}$ form a stable set, vertex z'_k ($1 \leq k \leq 2t + 2$) sees exactly z_{k-1}, z_k on $Q \cup X \cup Y \cup \{u, v\}$ (with $z_0 = x_i$ and $z_{2t+2} = x_{i+1}$).

Let $x_0 = u$ and $x_{r+1} = v$. If x_{i-1} sees y_{j+1}, then there is a strong path X', Y', Z between u and v with $X' \subseteq \{x_1, \ldots, x_{i-1}, y_{j+1}, \ldots, y_s\}, Y' \subseteq \{y_1, \ldots, y_j, x_{i+1}, \ldots, x_r\}$ such that $|X' \cup Y'| < |X \cup Y|$, a contradiction to the minimality of $|X \cup Y|$. So, x_{i-1} misses y_{j+1}. Similarly, y_{j-1} misses x_{i+1}. If x_{i-1} misses y_j and y_{j-1} misses x_i, then $X \cup Y \cup \{u\}$ contains a hole. So, there exists $z'_0 \in \{x_{i-1}, y_{j-1}\}$ that misses both x_{i+1} and y_{j+1} and sees both x_i and y_j. Similarly, there exists $z'_{2t+3} \in \{x_{i+1}, y_{j+1}\}$ that misses both x_i and y_j and sees both x_{i+1} and y_{j+1}. Write $z_{-1} = y_j$, $z_0 = x_i$, $z_{2t+2} = x_{i+1}$, $z_{2t+3} = y_{j+1}$. Now, the vertices $z_{-1}, z_0, z_1, z_2, z_3, z'_0, z'_1, z'_2, z'_3$ form the graph F_{23}, a contradiction. □

Thus, we may consider only parallel quadruples in which strong paths use no attachment of type 2. Even with this restriction, one can obtain many minimal forbidden subgraphs of rooted path graphs, and we can find no simple way to represent them as a finite number of infinite families, as the case for interval graphs, path graphs or directed path graphs.

There is no way to represent all types of strong paths, with attachment of type 1, that can link two vertices in a chordal graph. One can put many cliques of size four between $X \cup Y$ (each with an attachment of type 1). And, given such a strong path X, Y, Z

linking z_1 and z_2 and such a strong path X', Y', Z' linking z_3 and z_4 (all these vertices are distinct and there are no edges between the two parts), one can easily construct a graph which is minimally not a rooted path graph by adding a path v_1-$\cdots$-v_ℓ ($\ell \geq 1$) where v_1 sees all the vertices of $X \cup Y \cup Z$ and v_ℓ sees all the vertices of $X' \cup Y' \cup Z'$ (see Figure 9).

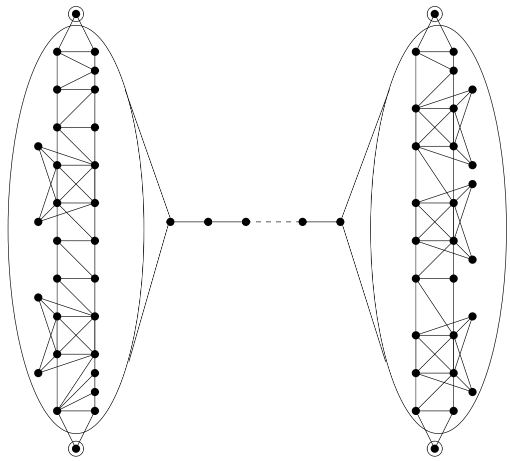


Figure 9: Example of graph which is minimally not a rooted path graph and which contains a weak asteroidal quadruple

And, there are many more different possibilities as the two strong paths may share some vertices, they may have some edges between them, the attachment of type 1 can see more than 3 vertices on the strong path, they can be used in more than one clique of size four, etc. In Figure 10, we give more examples of graphs which are minimally not rooted path graphs and which contain a weak asteroidal quadruple where strong paths are just a common neighbor. Even with this restriction, they are many variants and we give just a few.

8 Conclusion

We have defined particular asteroids on three and four vertices to obtain a characterization of directed path graphs and some partial results on rooted path graphs. The characterization of rooted path graphs that we conjecture will be a nice alternative to finding the list of all minimal forbidden induced subgraphs for this class.

One can also try to do similar work for path graphs. Path graphs are a superclass of directed path graphs that may contain some strong asteroidal triples (odd suns). Can one define a particular type of strong asteroidal triple that will give a nice characterization of path graphs?

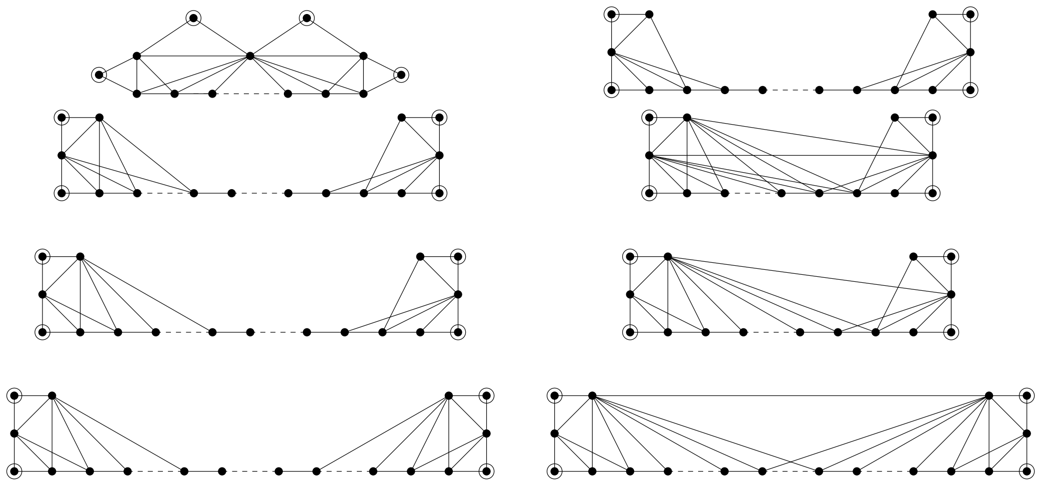


Figure 10: Examples of minimal forbidden induced subgraphs of rooted path graphs which contain a weak asteroidal quadruple

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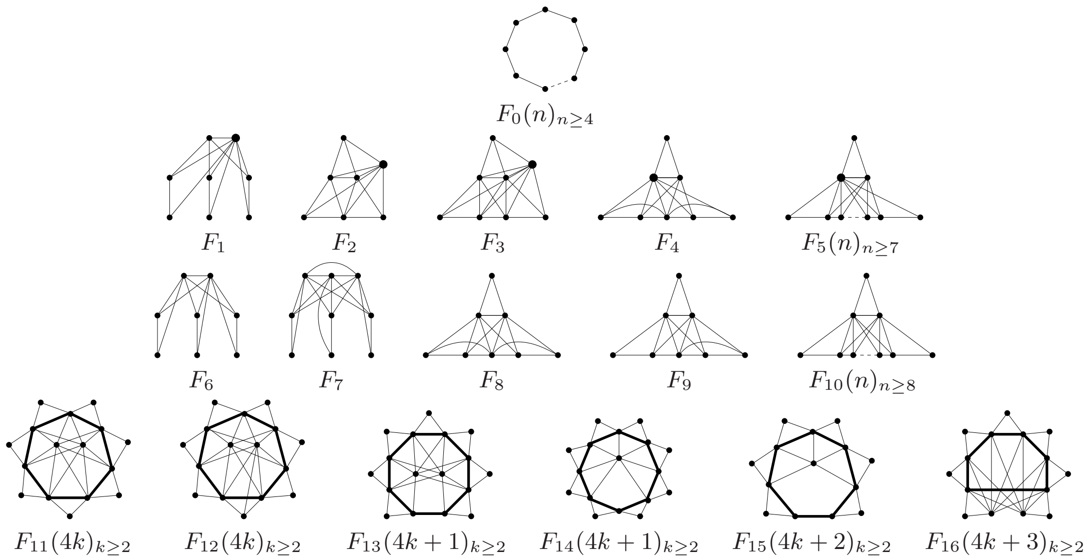

Figure 11: Minimal forbidden induced subgraphs for path graphs. (bold edges form a clique)

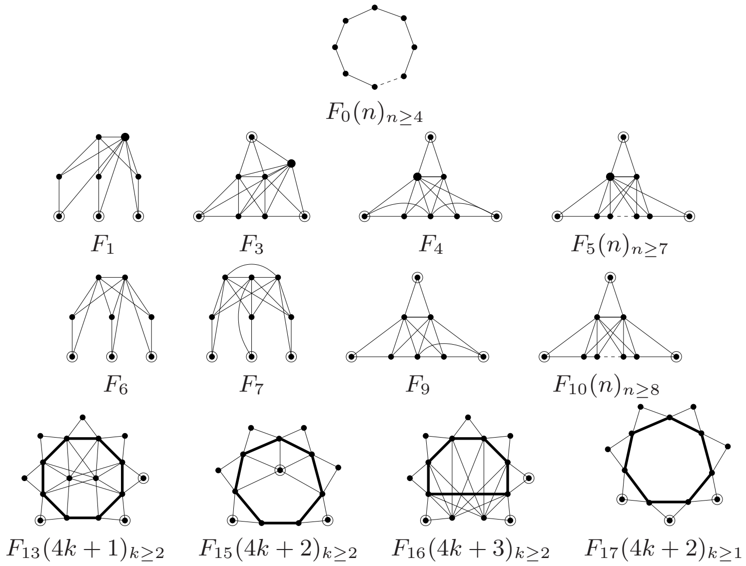

Figure 12: Minimal forbidden induced subgraphs for directed graphs. (bold edges form a clique)

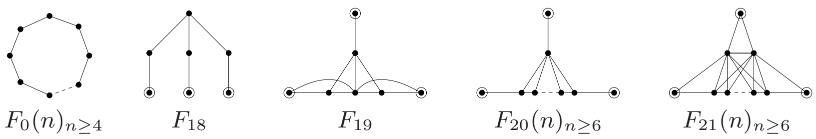

Figure 13: Minimal forbidden induced subgraphs for interval graphs.

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Responsables de la publication : Nadia Brauner et András Sebő
ISSN : 1298-020X