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Abstract

A Gallai-coloring of a complete graph is an edge coloring such that no triangle is colored with three distinct colors. Gallai-colorings occur in various contexts such as the theory of partially ordered sets (in Gallai's original paper) or information theory. Gallai-colorings extend 2-colorings of the edges of complete graphs. They actually turn out to be close to 2-colorings - without being trivial extensions.

Here we give a method to extend some results on 2-colorings to Gallai-colorings, among them known and new, easy and difficult results. The method works for *Gallai-extendible* families that include for example double stars and graphs of diameter at most d for $2 \leq d$, or complete bipartite graphs. It follows that every Gallai-colored K_n contains a monochromatic double star with at least $\frac{3n+1}{4}$ vertices, a monochromatic complete bipartite graph on at least $n/2$ vertices, monochromatic subgraphs of diameter two with at least $\frac{3n}{4}$ vertices, etc.

The generalizations are not automatic though, for instance a Gallai-colored complete graph does not necessarily contain a monochromatic star on $n/2$ vertices. It turns out that the extension is possible for graph classes closed under a simple operation called *equalization*.

We also investigate Ramsey numbers of graphs in Gallai-colorings with a given number of colors. For any graph H let $RG(r, H)$ be the minimum m such that in every Gallai-coloring of K_m with r colors, there is a monochromatic copy of H. We show that for fixed H, $RG(r, H)$ is exponential in r if H is not bipartite; linear in r if H is bipartite but not a star; constant (does not depend on r) if H is a star (and we determine its value).

1 Introduction

We consider edge colorings of complete graphs in which no triangle is colored with three distinct colors. In [20] such colorings were called Gallai partitions, in [16] the term Gallai colorings was used. The reason for this terminology stems from its close connection to results of Gallai on comparability graphs [14]. We will use the term *Gallai-coloring* and we assume that Gallai-colorings are colorings on complete graphs. It is useful to keep in mind a particular Gallai-coloring - sometimes called canonical coloring - where all color classes are stars ($V = [n]$ and for all $1 \leq i < j \leq n$ edge ij has color i.

More than just the term, the concept occurs again and again in relation of deep structural properties of fundamental objects. A main result in Gallai's original paper – translated to English and endowed by comments in [24] – can be reformulated in

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terms of Gallai-colorings. Basic results about comparability graphs can be equivalently discussed in terms of Gallai-colorings, as the theorem below shows. Further occurrences are related to generalizations of the perfect graph theorem [6], or applications in information theory [19].

The following theorem expresses the key property of Gallai-colorings. It is stated implicitly in [14] and appeared in various forms, [5], [6], [16]. The following formulation is from [16].

Theorem 1. *Any Gallai-coloring can be obtained by substituting complete graphs with Gallai-colorings into vertices of a* 2*-colored complete graph on at least two vertices.*

The substituted complete graphs are called *blocks* whereas the 2-colored complete graph into which we substitute is the *base graph.* Substitution in Theorem 1 means replacements of vertices of the base graph by Gallai-colored blocks so that all edges between replaced vertices keep their colors.

Theorem 1 is an important tool for proving results for Gallai-colorings. For example, it was used to extend Lovász's perfect graph theorem to Gallai-colorings, see [6], [20]. In [5] a more refined decomposition of Gallai-colorings was established. In this paper we focus on the following subjects:

- Extending 2-coloring results as black boxes
- Gallai colorings with fixed number of colors

1.1 Gallai-extension using black boxes

In [16] Ramsey type theorems for 2-colorings were extended to Gallai-colorings, using Theorem 1. Here we have a similar goal, but we accomplish it using a completely different method. Instead of extending the proofs of 2-coloring results, we define a property - we call it *Gallai-extendible* - of families of graphs that automatically carries over 2-coloring results to Gallai-colorings.

Definition. A family $\mathcal{F}$ of finite connected graphs is *Gallai-extendible* if it contains all stars and if for all $F \in \mathcal{F}$ and for all proper nonempty $U \subset V(F)$ the graph $F' = F'(U)$ defined as follows is also in $\mathcal{F}$:

- $V(F') = V(F)$
- $E(F') = E(F) \setminus \{uv : u, v \in U\}) \cup \{ux : u \in U, x \notin U, vx \in E(F) \text{ for some } v \in U\}$.

We will say that F' is the *equalization* of F in U. The conditions that Gallai-extendible families must contain only connected graphs and must contain all stars are somewhat technical. However, it seems that no application can really utilize more general definitions - and in the canonical Gallai coloring every color class is a star.

Our main result, Theorem 2, states that if every 2-colored K_n contains a monochromatic F of a certain order from a Gallai-extendible family then this remains true for Gallai-colorings: every Gallai-colored K_n also contains from the same family a monochromatic F' such that $|V(F')| \geq |V(F)|$.

Theorem 2. *Suppose that $\mathcal{F}$ is a Gallai-extendible family, and that there exists a function $f : \mathbf{N} \to \mathbf{N}$ such that for every n and for every 2-coloring of K_n there is a monochromatic $F \in \mathcal{F}$ with $|V(F)| \geq f(n)$.*

Then, for every n and every Gallai-coloring of K_n there exists a monochromatic $F' \in \mathcal{F}$ such that $|V(F')| \geq f(n)$ – with the same function f.

Moreover, such an F' exists in one of the colors used in the base-graph and also with no edge of F' within a block of the base graph.

The proof of Theorem 2 is in Section 2 together with several examples of Gallai-extendible families (Lemma 1). Applying Theorem 2 to these families, we get the following corollaries (the first two were known before, the others are new). If G is a graph, then H is called a spanning subgraph, if $V(H) = V(G)$. Applying Theorem 2 to the family of connected graphs we get

Corollary 1. *Every Gallai-colored complete graph contains a monochromatic spanning tree.*

For 2-colorings, Corollary 1 is the well-known remark of Erdős and Rado - a first exercise in graph theory. For Gallai-colorings it was proved by Bialostocki, Dierker and Voxman in [1]. Applying Theorem 2 to the family of graphs having a spanning tree of height at most two, we get

Corollary 2. *Every Gallai-colored complete graph contains a monochromatic spanning tree of height at most two.*

For 2-colorings Corollary 2 is due to [1], for Gallai-colorings it was proved in [16]. Applying Theorem 2 to the family of graphs with diameter at most three, we get

Corollary 3. *Every Gallai-colored complete graph contains a monochromatic spanning subgraph of diameter at most three.*

For 2-colorings Corollary 3 can be found in [1], [25], [26]. Applying Theorem 2 to the family of graphs with diameter at most two, we get

Corollary 4. *Every Gallai-colored K_n contains a monochromatic subgraph of diameter at most two with at least $\lceil \frac{3n}{4} \rceil$ vertices. This is best possible for every n.*

For 2-colorings, this is due to Erdős and Fowler, [9] (a weaker version with an easy proof is in [15]). The following construction ([9]) shows that Corollary 4 is sharp: consider a 2-coloring of K_4 with both color classes isomorphic to P_4. Then substitute nearly equal vertex sets into this coloring with a total of n vertices. (The colorings within the substituted parts can be arbitrary.) Applying Theorem 2 to the family of graphs containing a spanning double-star (two vertex disjoint stars joined by an edge), we get

Corollary 5. *Every Gallai-colored K_n contains a monochromatic double star with at least $\frac{3n+1}{4}$ vertices. This is asymptotically best possible.*

The 2-color version of Corollary 5 is (a special case of) a result in [17], it slightly extends a special case of a result in [8]: in every 2-coloring of K_n there are two points, v, w and a color, say red, such that the size of the union of the closed neighborhoods of v, w in red is at least $\frac{3n+1}{4}$. (The slight extension is that one can also guarantee that the edge vw is red.) Corollary 5 is asymptotically best possible, as shown by a standard random graph argument in [8]. Applying Theorem 2 to the family of graphs containing a spanning complete bipartite graph, we get

Corollary 6. *Every Gallai-colored K_n contains a monochromatic complete bipartite subgraph with at least $\lceil \frac{n+1}{2} \rceil$ vertices, and at least one more if n is congruent to -1 modulo 4.*

For 2-colorings Corollary 6 follows easily since there is a monochromatic star of the required size. However, for Gallai-colorings there are not always monochromatic stars with $\lceil \frac{n+1}{2} \rceil$ vertices - the largest monochromatic star has $\frac{2n}{5}$ vertices, see [16]. It is worth noting that Corollary 6 is best possible for every n. Paley graphs provide infinitely many examples, but there are simpler 2-colorings that do not contain monochromatic complete bipartite graphs larger than the size claimed in Theorem 6. Consider the vertex set as a regular n-gon and define the red graph by edges xy forming a diagonal of length at most k (if $n = 4k, 4k+1, 4k+2$) or at most $k+1$ (if $n = 4k+3$).

We conclude this part with some remarks on Gallai-extendible families. A *broom* is the union of a path and a star where the end-vertex of the path coincides with the central vertex of the star, and this is the only common vertex of the two. Burr [3] proved that every 2-colored complete graph has a monochromatic spanning broom. Gyárfás and Simonyi [16] extended Burr's theorem to Gallai-colorings. We can not reprove this result with Theorem 2 as a black box extension of Burr's theorem because brooms are not Gallai-extendible. However - and similar ideas might be useful in other potential applications of Theorem 2 - it is possible to combine a key element of Burr's proof with Gallai-extendable families (in our case with $\mathcal{F}_5$, i.e. graphs containing a spanning complete bipartite graph) to extend Burr's theorem to Gallai colorings.

1.2 Gallai colorings with given number of colors

As mentioned above, in canonical Gallai-colorings each color class is a star, thus Gallai-colorings do not necessarily contain any monochromatic H different from a star (apart from isolated vertices). However, we may define for any graph H a kind of restricted Ramsey number, $RG(r,H)$, the minimum m such that in every Gallai-coloring of K_m with r colors, there is a monochromatic copy of H.

It turns out that some classical Ramsey numbers whose order of magnitude seem hopelessly difficult to determine, behave nicely if we restrict ourselves to Gallai-colorings with r-colors. For example, the Ramsey number of a triangle in r-colorings, $R(r,K_3)$ is known to be between bounds far apart (c^r and $\lfloor er! \rfloor + 1$, see for example in [22]) but it is not hard to determine $RG(r,K_3)$ *exactly* as follows.

Theorem 3.

$$RG(r,K_3) = \begin{cases} 5^k+1 & \text{for } r=2k \\ 2\times 5^k+1 & \text{for } r=2k+1 \end{cases}$$

In fact - as we were informed by C. Magnant, [23] - Theorem 3 is due to Chung and Graham, [7]. Here we give a simpler proof, using Theorem 1.

It is worth noting that there are several "extremal" colorings for Theorem 3. For example, let G_1 be a black edge and let G_2 be the K_5 partitioned into a red and a blue pentagon. The graphs H_1, H_2 obtained by substituting G_1 (G_2) into vertices of G_2 (G_1) have essentially different 3-colorings and both are extremal for $r=3$ in Theorem 3.

Although one can easily determine some more exact values of $RG(r,H)$ for small graphs H, we conclude with the following two theorems that determine its order of magnitude.

Theorem 4. *Assume that H is a fixed graph without isolated vertices. Then $RG(r,H)$ is exponential in r if H is not bipartite and linear in r if H is bipartite and not a star.*

Theorem 5. *If $H = K_{1,p}$ is a star and $r \geq 3$ then $RG(r,H) = \frac{5p-1}{2}$ for odd p, $RG(r,H) = \frac{5p}{2} - 3$ for even p.*

For completeness of the star case, notice that for $H = K_{1,p}$ we have trivially $RG(1,H) = R(1,H) = p+1$ and $RG(2,H) = R(2,H)$ can be determined easily ($2p-1$ for even p and $2p$ for odd p, [18]). It is also worth noting that while $RG(r,H)$ is constant (does not depend on r), $R(r,H)$ is linear in r (and in p), see [4].

A Gallai-coloring can be also viewed as an anti-Ramsey coloring for C_3 - anti-Ramsey colorings for a graph H have been introduced in [10]. This direction has a large literature that we do not touch here. Moreover, Gallai-colorings are also connected to so- called mixed Ramsey numbers, where the aim is to find either a multicolored graph G (in our case a triangle) or a monochromatic graph H. We are aware of some papers in preparation that determine exact values of $RG(r,H)$. Faudree, Gould, Jacobson and Magnant [11] determined the value of $RG(r,H)$ for many bipartite graphs H. Fujita [12] proved that $RG(r,C_5) = 2^{r+1}+1$; Fujita and Magnant [13] extended Gallai-colorings to colorings without a rainbow S_3^+, a triangle with a pendant edge.

2 Gallai-extendible families - proof of Theorem 2

We denote by $\text{dist}_H(u,v)$ the number of edges in a shortest path of H between $u,v \in V(H)$.

Lemma 1. *The following families are Gallai-extendible:*

- $\mathcal{F}_1$*, the family of connected graphs;*
- $\mathcal{F}_2(d)$*, the family of graphs having a spanning tree of height at most* d*, for any* $d \geq 2$ *– equivalently a root* $x \in V(F)$ *such that dist*$(x,v) \leq d$ *for all* $v \in V(F)$*;*
- $\mathcal{F}_3(d)$*, the family of graphs with diameter at most* d *for any* $d \geq 2$*;*
- $\mathcal{F}_4$*, the family of graphs having a spanning double-star – equivalently, two adjacent vertices forming a dominating set;*
- $\mathcal{F}_5$*, the family of graphs containing a spanning complete bipartite graph, that is, the family of graphs* F *so that* $V(F)$ *can be partitioned into two nonempty sets* A *and* B *so that* $ab \in E(F)$ *for all* $a \in A$*,* $b \in B$*;*

Proof. We prove for all these families $\mathcal{F}$ and every $F \in \mathcal{F}$, and for all proper nonempty $U \subseteq V(F)$ (or $U \in \mathcal{U}_F$) that the graph F' we get after equalizing in U is still in $\mathcal{F}$. Since the five families we consider are closed under the addition of edges and it is immediate from the definition that equalization is a monotonous operation, that is, $F_1 \subseteq F_2$ implies $F_1' \subseteq F_2'$, it is sufficient to prove $F' \in \mathcal{F}$ for minimal elements $F \in \mathcal{F}$. Whenever it is comfortable to exploit this fact we will do it: for instance when checking the statement for $\mathcal{F}_2$ or $\mathcal{F}_4$, F can be chosen to be a tree of height at most two, or a double-star.

For $\mathcal{F}_1$ the statement is immediate noting that the connectivity of F implies that whenever an edge $e = xy \in E(F), F \in \mathcal{F}_1$ disappears, there exists a path of length 2 in F' between its endpoints.

For $\mathcal{F} = \mathcal{F}_2(d)$ or $\mathcal{F} = \mathcal{F}_3(d)$ the following claim will provide the statement:

Claim: For $u, v \in V(F)$, $uv \in E(F)$ we have $\text{dist}_{F'}(u, v) \leq 2$, and if $uv \notin E(F)$ then $\text{dist}_{F'}(u, v) \leq \text{dist}_F(u, v)$

Indeed, if $uv \in E(F)$, then either at least one of u and v is not in U, and then $uv \in E(F')$, or $u, v \in U$, and then - from the connectivity of F and the fact that U is a proper subset of $V(F)$ - they have a common F'-neighbor. The first part is proved.

To prove the second part, let P be a shortest path in F between u and v, $|E(P)| \geq 2$. Then P can be subdivided to subpaths induced by U and other subpaths (there must be others, since otherwise replace P by a two-path from u to v). Define the path P' in F' between u and v by replacing the subpaths in U by an arbitrary vertex in the subpath - in the special case when u or v is on the subpath, replace it by u or v. Since all vertices of U have the same neighbors outside U, P' will indeed be a path in F', and $|E(P')| \leq |E(P)|$, as claimed.

Now if $F \in \mathcal{F}_2(d)$ $(d \geq 2)$, apply the claim to the root x, and all other vertices $v \in V(F)$ to get that $F' \in \mathcal{F}_2(d)$. Similarly, if $F \in \mathcal{F}_3(d)$, apply the claim to all pairs $u, v \in V(F)$.

If $F \in \mathcal{F}_4$, let $xy \in E(F)$ be such that $V(F)$ consists of neighbors of x and neighbors of y. If neither x nor y are in U, no edge is deleted at equalization and there is nothing to prove. Similarly, if exactly one of them is in U, say $x \in U$, $y \notin U$, then $xy \in F$ implies that y is adjacent in F' with every vertex in U, and the vertices that are not in U remain neighbors of x or y in F' as well.

It remains to check $F' \in \mathcal{F}_4$ if both $x, y \in U$. This is also easy, because every vertex of F is adjacent to at least one of x and y, and therefore in F' every vertex of U is adjacent to every vertex in $V(F) \setminus U$. We are then done, because a complete bipartite graph contains a spanning double-star.

Let $F \in \mathcal{F}_5$. If one of the two classes, say A is disjoint of U, $F \subseteq F'$, so the statement is obvious. If now U meets both A and B in a vertex a and b respectively, we are also done, since all $A \cup B$ is F-adjacent with either a or b, so all vertices of

$U \cap (A \cup B)$ are F'-adjacent with all vertices of $(A \cup B) \setminus U$, and both of these sets are nonempty, finishing the proof for this class. $\square$

Proof of Theorem 2. Suppose that $\mathcal{F}$ is a Gallai-extendible family and c is a Gallai-coloring of K_n. By Theorem 1, c can be obtained by substituting Gallai-colored complete graphs into the vertices $\{v_1, v_2, \ldots, v_k\}$ of a base graph B with a red-blue coloring , $k \geq 2$. Suppose that B is connected in red (in fact, we shall use only that B has no isolated vertex in red). The vertex sets of the substituted complete graphs give a partition $\mathcal{U}$ on $V(K_n)$.

Let c' be the 2-coloring of K_n obtained from c by recoloring all edges within all blocks of the partition $\mathcal{U}$ to the red color. In the coloring c', by the assumption of Theorem 2, K_n has a subgraph $F \in \mathcal{F}$ with $|V(F)| \geq f(n)$, such that F is monochromatic in c'. If F is blue then F is a monochromatic subgraph in c as well and the proof is finished.

Thus we may assume that $F \subseteq E_{c'}(\text{red})$ (the red edges in c'). If $V(F) \subseteq U$ for some $U \in \mathcal{U}$ then - using that B has no isolated vertex in the red color - we can select a star S in K_n such that S is red in c, its center $v \notin U$ and its leaf set is U. Now $S \in \mathcal{F}$ (because $\mathcal{F}$ contains all stars) and $|V(S)| > |V(F)|$, finishing the proof.

Thus we may assume that $V(F)$ is not a subset of any block of $\mathcal{U}$. Now equalize F in the blocks of $\mathcal{U}$ one after the other. Since F is connected and $V(F)$ is not a subset of some block, eventually all recolored edges will be deleted during the equalizations. We claim that the graph F' resulting from the equalization process is a subgraph of $E_c(\text{red})$. Indeed, equalization adds an edge ux ($u \in U$) only if $x \notin U$, and there exists $v \in U$, $vx \in E(F)$. Since $E(F) \subseteq E_{c'}(\text{red})$, and vx is not a recolored edge, $vx \in E_c(\text{red})$ follows. Since every block sends only edges of one and the same color to every vertex, $ux \in E_c(\text{red})$ as well, confirming the claim.

Since $\mathcal{F}$ is Gallai-extendible, $F' \in \mathcal{F}$, and clearly $|V(F')| \geq |V(F)| \geq f(n)$. Now the proof is finished (the extra property stated about F' is obvious). $\square$

3 Proof of Theorems 3,4,5.

Proof of Theorem 3: Let $f(r)$ denote the function one less than the claimed value of $RG(r, K_3)$. Observe that

$$f(r) \geq 2f(r-1) \tag{1}$$

for $r \geq 2$ with equality for odd r, and

$$f(r) = 5f(r-2) \tag{2}$$

for $r \geq 3$.

To show that $RG(r, K_3) > f(r)$ let G_1 be a 1-colored K_2 and let G_2 be a 2-colored K_5 with both colors forming a pentagon. Recursively construct G_r for odd $r \geq 3$ by substituting two identically colored G_{r-1}'s into the two vertices of G_1 (colored with a different color). Similarly, for even $r \geq 4$, let G_r be defined by substituting five identically colored G_{r-2}'s into the vertices of G_2 (colored with two different colors). The r-coloring defined on G_r is a Gallai-coloring, clearly has $f(r)$ vertices and contains no monochromatic triangles.

We prove by induction that if a Gallai-coloring of K with r-colors and without monochromatic triangles is given then $|V(K)| \leq f(r)$. Using Theorem 1, the coloring of K can be obtained by substitution into a 2-colored nontrivial base graph B. In our case clearly $2 \leq |V(B)| \leq 5$.

Case 1: $|V(B)| = 2$. Since there are no monochromatic triangles, the graphs substituted can not contain any edge colored with the color of the base edge, therefore, by induction, they have at most $f(r-1)$ vertices. Thus

$$|V(K)| \leq 2f(r-1) \leq f(r)$$

using (1).

Case 2: $|V(B)| = 3$. The base graph has no monochromatic triangle so it has an edge b_1b_2 whose color is used only once (as a color on a base edge). Then the graphs substituted into b_1, b_2 must be colored with at most $r-2$ colors and the graph substituted into the third vertex must be colored with at most $r-1$ colors. Thus

$$|V(K)| \leq 2f(r-2) + f(r-1) \leq f(r-1) + f(r-1) = 2f(r-1) \leq f(r)$$

using (1) twice.

Case 3: $4 \leq |V(B)| \leq 5$. The base graph has no monochromatic triangle so each vertex in the base is incident to edges of both colors. Therefore

$$|V(K)| \leq |V(B)|f(r-2) \leq 5f(r-2) = f(r)$$

using (2). □

Proof of Theorem 4: First we give an upper bound on $RG(r, H)$ that is exponential in r by showing $RG(r, H) \leq t^{(n-1)r+1}$ where $t = R(2, H) - 1$ and $n = |V(H)|$. We shall assume that $|V(H)| \geq 3$ therefore $n \geq 3, t \geq 2$. Suppose indirectly that a Gallai-coloring with r colors is given on K, $|V(K)| \geq t^{(n-1)r+1}$ but there is no monochromatic H. The base graph B of this coloring has no monochromatic H therefore $|V(B)| \leq R(2, H) - 1 = t$. This implies that some of the graphs, say G_1, substituted into B has at least $t^{(n-1)r}$ vertices. Let v_1 be an arbitrary vertex of K not in $V(G_1)$. Note that every edge from v_1 to $V(G_1)$ has the same color. Iterating this process with G_1 in the role of K, one can define a sequence of vertices $v_1, v_2, \ldots, v_{(n-1)r+1}$ such that for every fixed i and $j > i$ the colors of the edges v_i, v_j are the same. By the pigeonhole principle there is a subsequence of n vertices spanning a monochromatic complete subgraph $K_n \subset K$ and clearly H is a monochromatic subgraph of K_n - a contradiction. Thus, for any - in particular non-bipartite - H we proved an upper bound exponential in r.

For a bipartite H assume that both color classes of H have at most n vertices. We show that $RG(r,H) \le pt(n-1)$, where $p=(n-1)r+2$ (and t is as defined earlier), providing an upper bound linear in r. Indeed, suppose indirectly that a Gallai-coloring with r colors is given on K, $|V(K)| \ge pt(n-1)$ but there is no monochromatic H. The base graph of the Gallai-coloring has at most t vertices, otherwise we have a monochromatic H. Applying the same argument as in the previous paragraph, we find that there is a graph G_1, substituted to some vertex of the base graph, such that $|V(G_1)| \ge \frac{|V(K)|}{t} \ge p(n-1)$. If $|V(K) \setminus V(G_1)| \ge 2n-1$ then - by the pigeonhole principle - we can select $X \subset V(K) \setminus V(G_1)$ so that $|X| = n$ and $[X, V(G_1)]$ is a monochromatic complete bipartite graph - this graph contains H and the proof is finished. We conclude that $|V(G_1)| \ge pt(n-1) - 2(n-1) = (pt-2)(n-1)$. Select $v_1 \in V(K) \setminus V(G_1)$ and iterate the argument: into some vertex of the base graph of the Gallai-coloring on G_1 a graph G_2 is substituted with at least $\frac{|V(G_1)|}{t} \ge (p-1)(n-1)$ vertices. Selecting $v_2 \in V(G_1) \setminus V(G_2)$ we continue until $T = \{v_1, v_2, \dots, v_{p-1}\}$ is defined. There is still at least $2(n-1) > n$ vertices in G_{p-1} thus selecting $Y \subset V(G_{p-1})$ with $|Y| = n$, we have a complete bipartite graph $[Y,T]$ such that from each $v \in T$ all edges from Y to v are colored with the same color. Since $|T| = p-1 = (n-1)r+1$, by the pigeonhole principle there is $Z \subset T$ such that $|Z| = n$ and $[Y,Z]$ is a monochromatic complete bipartite graph which obviously contains a monochromatic H - a contradiction. Thus, for bipartite H we have an upper bound linear in r.

Lower bounds of the same order of magnitude can be easily given. For a non-bipartite H it is obvious that $RG(r,H) > 2^r$ because we can easily define a suitable Gallai-coloring with r colors by repeatedly joining with a new color two identically colored complete graphs of the same size.

If H is bipartite and not a star, it contains two *independent*, that is, vertex-disjoint edges. Then we have $RG(r,H) > r+1$ because the canonical Gallai-coloring of K_{r+1} with r colors (where color class i is a star with i edges) does not have a monochromatic H. □

Proof of Theorem 5: Assume $H = K_{1,p}, r \ge 3$. We use a construction and a result from [16]. To see that the claimed values of $RG(r,H)$ can not be lowered, let C be a K_5 colored with red and blue so that both color classes form a pentagon. For odd p substitute a green $K_{\frac{p-1}{2}}$ to each vertex of C. For even p substitute $K_{\frac{p}{2}}$ into one vertex of C and $K_{\frac{p}{2}-1}$ to the other four vertices of C. The claimed upper bound for $RG(r,H)$ follows immediately from the following result of [16]: any Gallai-coloring of K contains a monochromatic star with at least $\frac{2|V(K)|}{5}$ edges. □

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