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# Generic algorithms for some decision problems on fasciagraphs and rotagraphs

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# Generic algorithms for some decision problems on fasciagraphs and rotagraphs

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**Abstract**

A fasciagraph consists in a sequence of copies of the same graph, each copy being linked to the next one according to a regular scheme. More precisely a fasciagraph is characterized by an integer $n$ (the number of copies or fibers) and a mixed graph $M$. In a rotagraph, the last copy is also linked to the first one. In the literature, similar methods were used to address various problems on rotagraphs and fasciagraphs. The goal of our work is to define a class of decision problems for which this kind of methods works. For this purpose, we introduce the notion of pseudo-$d$-local $q$-properties of fasciagraphs and rotagraphs. For a mixed graph $M$ and a pseudo-$d$-local $q$-property $\mathcal{P}$, we propose a generic algorithm for rotagraphs (resp. fasciagraphs) that computes in constant time a closed formula as a boolean function $\tau$ (resp. $\varphi$) on $\mathbb{N}$. For all $n \in \mathbb{N}$ $\tau(n) = 1$ (resp. $\varphi(n) = 1$) if and only if the rotagraph (resp. fasciagraph) of length $n$ based on $M$ satisfies $\mathcal{P}$.

## 1 Introduction

In algorithmic graph theory, operational research, and combinatorial optimization, most of the studied problems are $\mathcal{NP}$-hard problems. Nevertheless, some of them become easier (polynomial) if their study is restricted to particular subclasses of graphs (trees, planar graphs, graphs of bounded tree-width, etc.). Among those classes of graphs, grids are widely studied in the literature, especially to represent interconnexion models of multiprocessors in VLSI systems (see the survey [11]). Fasciagraphs and rotagraphs have been introduced to model different problems of polymer chemistry (see [2], [5], [7]). These classes of graphs generalize grids.

In the literature, several $\mathcal{NP}$-Hard problems are shown to be much easier (sub-linear) when restricted on rotagraphs and fasciagraphs, or subclasses of them. The method shared by these papers is the following. Given a mixed graph $M$, a closed formula is computed by looking for paths or circuits of given length in an auxiliary directed graph. Then this formula enables to solve the problem for the fasciagraph (or rotagraph) based on $M$ of length $n$ for any $n$ (see [1], [3], [4], [5], [6], [7], [8], [9], [10], [12], [13], [14]).

However in each of these papers, a proof is given to explain the validity of this approach to address the considered problems on fasciagraphs or rotagraphs.

The goal of our work is to give a theoretical framework to characterize the class of problems for which this method can be used. An attempt to give such

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a characterization is made in [8], but several problems are not concerned even though the method could be used to address them. Moreover the characterization is not correct, we give a counter-example for a part of it in the Appendix. We address here decision problems by defining the notion of pseudo-$d$-local $q$-property. In a next paper, we will provide a characterization for optimization problems.

This paper is organized as follows: Section 2 is dedicated to the concepts and definitions which are needed for the proper understanding of the different results. In Section 3 we formally describe the class of problems we can deal with and in Section 4 we explain our algorithm in the case of rotagraphs. Sections 5 and 6 follow the same pattern as Sections 3 and 4 but for problems on fasciagraphs. Section 7 is dedicated to a summary of the key results. Finally we present in the Appendix a counter-example to an assertion in [8].

## 2 Definitions about rotagraphs and fasciagraphs

We give the following definitions that we will use in the rest of the paper:

**Definition 1.** *A mixed graph $M$ is a triple $(V, E, A)$ in which $V$ is a set of vertices, $E$ is a set of 2-elements subsets of $V$ and $A$ is a subset of $V \times V$. The elements of $E$ are called edges, we denote an edge $\{u, v\}$ by $uv$ or $vu$. The elements of $A$ are called arcs. (see Figure 1)*

**Definition 2.** *A directed graph is a mixed graph $G = (V, E, A)$ where $E = \emptyset$. We denote this graph by $\vec{G} = (V, A)$.*

**Definition 3.** *An undirected graph is a mixed graph $G = (V, E, A)$ where $A = \emptyset$. We denote this graph by $G = (V, E)$.*

In the following, the word "graph" will refer to an undirected graph, unless other specification.

**Definition 4.** *Let $G = (V, E)$ be a graph. A path of length $k$ in $G$ is a sequence of vertices $v_1, ..., v_{k+1}$ such that $v_i v_{i+1} \in E$ for all $i \in \{1, ..., k\}$.*

**Definition 5.** *Let $\vec{G} = (V, A)$ be a directed graph. A directed path of length $k$ in $\vec{G}$ is a sequence of vertices $v_1, ..., v_{k+1}$ such that $(v_i, v_{i+1}) \in A$ for all $i \in \{1, ..., k\}$.*

**Definition 6.** *Let $\vec{G} = (V, A)$ be a directed graph. A directed circuit of length $k$ in $\vec{G}$ is a sequence of vertices $v_1, ..., v_k, v_1$ such that $(v_i, v_{i+1}) \in A$ for all $i \in \{1, ..., k-1\}$, and $(v_k, v_1) \in A$.*

More generally, given a mixed graph $M$, the set of vertices of $M$ will be denoted by $V(M)$, the set of edges of $M$ will be denoted by $E(M)$, the set of arcs of $M$ will be denoted by $A(M)$.

**Definition 7.** *Let $\vec{G} = (V, A)$ be a directed graph whose vertices are labeled $v_1, v_2, ..., v_n$. The adjacency matrix of $\vec{G}$ is the binary matrix of size $|V| \times |V|$, denoted by $\Pi^{\vec{G}}$, or $\Pi$ if there is no ambiguity, and such that $\Pi^{\vec{G}}_{ij} = 1$ if and only if $(v_i, v_j) \in A$ $(i, j \leq |V|)$.*

**Definition 8.** *Let $G = (V, E)$ be an undirected graph and $u$ be a vertex of $G$. The neighborhood of $u$ in $G$, denoted by $\mathcal{N}_G(u)$, or $\mathcal{N}(u)$ if there is no ambiguity, is the set of vertices of $G$ linked to $u$ by an edge: $v \in \mathcal{N}_G(u) \Leftrightarrow uv \in E$.*

**Definition 9.** *Let $G = (V, E)$ be an undirected graph and $u$ a vertex of $G$. The closed neighborhood of $u$ in $G$, denoted by $\mathcal{N}_G[u]$ is equal to $\mathcal{N}_G(u) \cup \{u\}$.*

**Definition 10.** *Let $G = (V, E)$ be a graph and $D \subseteq V$. A vertex $x$ is said to be uniquely-dominated by $D$ if its closed neighborhood contains a single vertex of $D$.*

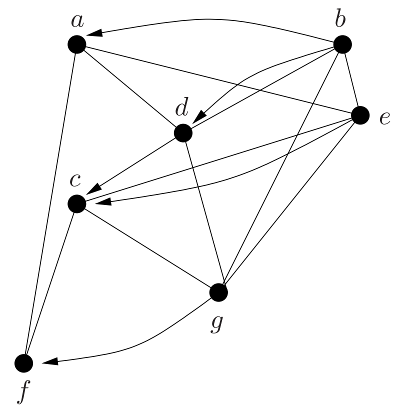


Figure 1: Example of a mixed graph $M$.

**Definition 11.** *Given a mixed graph $M = (V, E, A)$ and an integer $n \geq 2$, the fasciagraph $\psi_n(M)$ is the undirected graph defined in the following way:*

- $V(\psi_n(M)) = \bigcup_{i=1}^{n} V_i$ *where* $V_i = \{v_i | v \in V\}$ *for* $i \in \{1, ..., n\}$,
- $E(\psi_n(M)) = \bigcup_{i=1}^{n} E_i \cup \bigcup_{j=1}^{n-1} A_j$ *where* $E_i = \{u_i v_i | uv \in E\}$ *for* $i \in \{1, ..., n\}$ *and* $A_j = \{u_j v_{j+1} | (u, v) \in A\}$ *for* $j \in \{1, ..., n-1\}$.

*We will say that a graph $F$ is a fasciagraph if there exists an integer $n \geq 2$ and a mixed graph $M$ such that $F = \psi_n(M)$ (see Figure 2).*

**Remark 1.** *The fasciagraph $\psi_n(M)$ is made up of $n$ copies of the undirected graph $(V, E)$, connected according to a scheme defined by the set of arcs $A$. In particular, for all $i \in \{1, ..., n\}$, the graph $M_i = (V_i, E_i)$ is isomorphic to the graph $(V, E)$.*

**Remark 2.** *Fasciagraphs generalize grids. Indeed, the graph associated to a grid of size $m \times n$ is isomorphic to the fasciagraph $\psi_m(M)$ where $M = (V, E, A)$ with $V = \{v_1, ..., v_n\}$, $E = \{v_i v_{i+1} | i \in \{1, ..., n-1\}\}$, and $A = \{(v_i, v_i) | i \in \{1, ..., n\}\}$.*

**Definition 12.** *Given a mixed graph $M = (V, E, A)$, and an integer $n \geq 2$, the rotagraph $\omega_n(M)$ is the undirected graph defined in the following way:*

- $V(\omega_n(M)) = V(\psi_n(M))$,

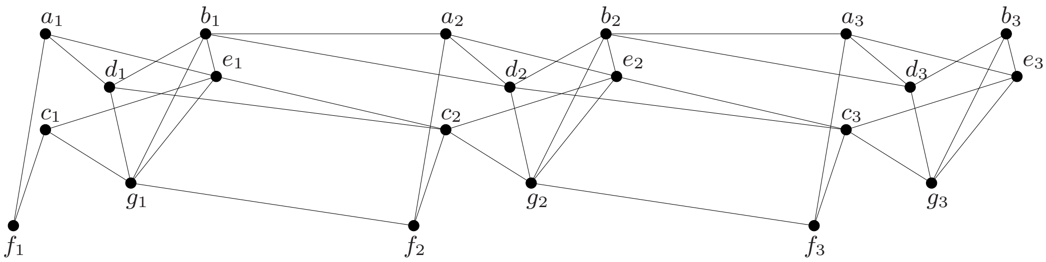


Figure 2: The fasciagraph $\psi_3(M)$ for the mixed graph $M$ of Figure 1.

- $E(\omega_n(M)) = E(\psi_n(M)) \cup \{u_n v_1 | (u, v) \in A\}$.

*We will say that a graph $R$ is a rotagraph if there exists an integer $n \geq 2$ and a mixed graph $M$ such that $R = \omega_n(M)$ (see Figure 3).*

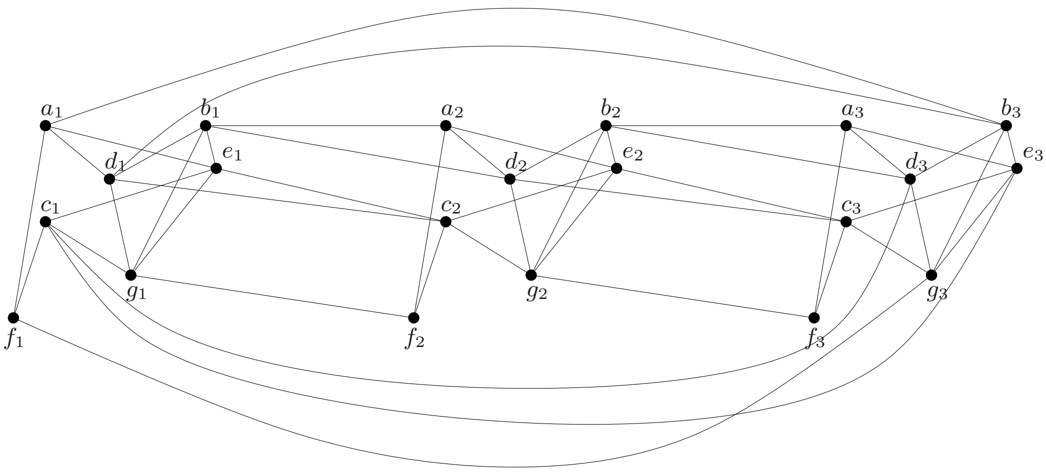


Figure 3: The rotagraph $\omega_3(M)$ where $M$ is the mixed graph of Figure 1.

**Remark 3.** *In other words, given a mixed graph $M = (V, E, A)$ and an integer $n$, the rotagraph $\omega_n(M)$ is isomorphic to the graph obtained by adding to $\psi_n(M)$ a set of edges from the last fiber to the first one according to the scheme defined by $A$.*

**Definition 13.** *Given a rotagraph $\omega_n(M)$, or a fasciagraph $\psi_n(M)$, the undirected graph $M_i = (V_i, E_i)$ for $i \in \{1, ..., n\}$ is called the $i$-th fiber of this rotagraph or fasciagraph.*

## 3 Decision problems and $q$-properties of rotagraphs

**Definition 14.** *Given a graph $G = (V, E)$, we define a $q$-labeling as follows:*

- *a $q$-labeling of $G$ is a function $f : V \cup E \rightarrow \{1, ..., q\}$,*

- *a q-labeling of the vertices of G is a function* $f : V \rightarrow \{1, ..., q\}$*,*
- *a q-labeling of the edges of G is a function* $f : E \rightarrow \{1, ..., q\}$*.*

*We denote by* $\mathcal{F}_q(G)$ *the set of q-labelings of G.*

**Definition 15.** *Let* $\mathcal{P}$ *be a property of graphs. We will say that* $\mathcal{P}$ *is a q-property if there exists a property* $\mathcal{P}'$ *on q-labelings of graphs such that, for every graph G:*

*G has the property* $\mathcal{P}$ *if and only if there exists* $f \in \mathcal{F}_q(G)$ *such that f has the property* $\mathcal{P}'$*.*

*Then we will say that* $\mathcal{P}'$ *is the characteristic property of labelings associated to* $\mathcal{P}$*.*

In the rest of the paper, the $q$-properties that we use, will be given directly as their characteristic labeling property.

Substituting in the above definition the word "graph" by the word "rotagraph" (resp. "fasciagraph"), we obtain the definition of a $q$-property of rotagraphs (resp. of fasciagraphs).

**Definition 16.** *Given a q-property* $\mathcal{P}$*, we denote by* $\mathcal{P}(G) \subseteq \mathcal{F}_q(G)$ *the set of q-labelings of G satisfying the characteristic property of* $\mathcal{P}$*. In other words, G has the q-property* $\mathcal{P}$ *if and only if there exists* $f \in \mathcal{P}(G)$*.*

**Examples**

- The property "to be $k$-colorable": a graph $G$ is said to be $k$-colorable if its vertices can be colored using at most $k$ colors avoiding adjacency between vertices of the same color. We notice that this property, denoted by $\mathcal{C}_k$ is a $k$-property. Indeed, it corresponds to a $k$-labeling of the vertices of $G$ such that no two adjacent vertices are labeled the same.
- The property "to contain a perfect dominating set": a graph $G = (V, E)$ has a perfect dominating set if there exists $W \subseteq V$ such that $W$ is an independent set, and every vertex out of $W$ has exactly one neighbor in $W$. We notice that this property, denoted by $\mathcal{D}_2$, is a 2-property. Indeed it is characterized by 2-labelings of the vertices of $G$ satisfying the following property $\mathcal{D}'_2$: each vertex is uniquely-dominated by the set of vertices of label 1.
- The property "to have a total $q$-coloring bounded by $b$" (for some integers $q$ and $b$): a fasciagraph $\psi_n(M)$ has a total $q$-coloring bounded by $b$ if there exists a $q$-labeling of the vertices and edges of $\psi_n(M)$ such that:
  - adjacent vertices have different labels,
  - adjacent edges have different labels,
  - incident vertex and edge have different labels,
  - for each fiber, the sum of the labels of its vertices is less than or equal to $b$.

**Remark 4.** *The two first examples above are properties of graphs, whereas the third one is a property of fasciagraphs since it uses the notion of fiber.*

*We also notice that for the two first examples, there are no constraints for the label on the edges. For that kind of q-properties, $\mathcal{F}_q(G)$ could be restricted to be the set of q-labelings of the vertices of G.*

*Similarly for q-properties characterized by q-labelings of edges, one could restrict $\mathcal{F}_q(G)$ to be the set of q-labelings of the edges of G.*

*All the results below are still valid with these alternative definitions of $\mathcal{F}_q(G)$.*

**Definition 17** (Partial $q$-labeling of a fasciagraph $q$-labeling)**.**

*Let $M = (V, E, A)$ be a mixed graph, $n \geq 2$, $k \geq 1$ and $f \in \mathcal{F}_q(\psi_n(M))$. We denote by $f_{i,k}$ the partial q-labeling of f which is the restriction of f on the k consecutive fibers $M_i, M_{i+1}, ..., M_{i+k-1}$, $i \in \{1, ..., n-k+1\}$. So $f_{i,k} \in \mathcal{F}_q(\psi_k(M))$.*

**Definition 18** (Partial $q$-labeling of a rotagraph $q$-labeling)**.**

*Let $M = (V, E, A)$ be a mixed graph, $n \geq 2$, $1 \leq k \leq n-1$, and $f \in \mathcal{F}_q(\omega_n(M))$. We denote by $f_{i,k}$ the partial q-labeling of f which is the restriction of f to the k consecutive fibers from fiber $M_i$, ($i \in \{1, ..., n\}$). If $i + k - 1 \leq n$, then the consecutive fibers will be $M_i, ..., M_{i+k-1}$. If $i + k - 1 > n$, then the consecutive fibers will be $M_i, ..., M_n, M_1, ..., M_{i+k-1-n}$. So $f_{i,k} \in \mathcal{F}_q(\psi_k(M))$.*

**Definition 19.** *Given an integer $d \geq 2$, $f \in \psi_n(M)$ and $g \in \psi_{n'}(M)$ such that $n, n' > d$ and $f_{n-d+1,d} = g_{1,d}$, the result of the d-concatenation of f and g denoted by $f \triangleright_d g$ is the q-labeling $h \in \mathcal{F}_q(\psi_{n+n'-d}(M))$ such that $h_{1,n} = f$ and $h_{n+1,n'-d} = g_{d,n'-d}$ (see Figure 4). If there is no ambiguity, we will write $\triangleright$ rather than $\triangleright_d$.*

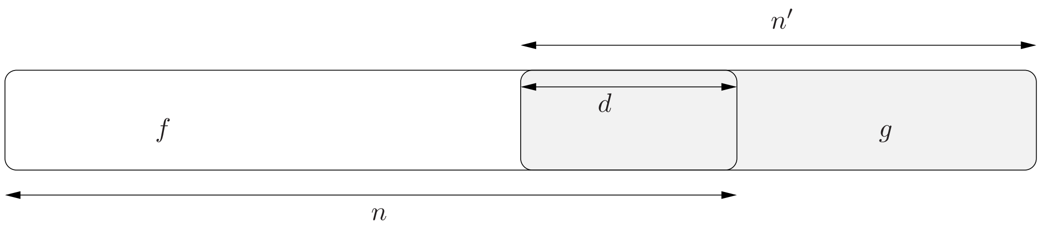


Figure 4: Scheme of $f \triangleright_d g$ on two $q$-labelings $f \in \psi_n(M)$ and $g \in \psi_{n'}(M)$

In [8] Klavžar and Vesel define two notions of heredity and $d$-locality of $q$-properties (see the Appendix). These notions exclude several classical properties such as dominating properties (perfect domination, identifying codes...). In order to cover a wide range of problems, including in particular perfect domination, identifying codes, (2,1)-labelings, etc., we define a different notion of locality: the pseudo-$d$-locality.

**Definition 20** (Pseudo-$d$-locality for rotagraphs)**.** *For an integer $d \geq 2$, a q-property $\mathcal{P}_q$ of rotagraphs is said to be pseudo-d-local if there exists a q-property $\mathcal{P}_q^{loc}$ on fasciagraphs of length d such that for every mixed graph $M = (V, E, A)$, for every integer $n > d$, and for every q-labeling $f \in \mathcal{F}_q(\omega_n(M))$,*

$$f \in \mathcal{P}_q(\omega_n(M)) \Leftrightarrow f_{i,d} \in \mathcal{P}_q^{loc}(\psi_d(M)) \text{ for every } i \in \{1, ..., n\}$$

**Example** Let us consider the 2-property $\mathcal{D}_2$ of perfect domination defined in the previous paragraph. This property is a pseudo-3-local 2-property. Indeed, for every mixed graph $M = (V, E, A)$ and for every 2-labeling $f \in \mathcal{F}_2(\psi_3(M))$, we define the 2-property $\mathcal{D}_2^{loc}$ as follows:

$f \in \mathcal{D}_2^{loc}(\psi_3(M))$ if and only if $f$ is a $q$-labeling of the vertices of $\psi_3(M)$ such that $\forall u \in V(M_2)$, $u$ is uniquely-dominated by $f^{-1}(1)$.

In other words, a 2-labeling $f$ of a rotagraph $\omega_n(M)$ ($n \geq 4$) corresponds to a perfect dominating set if and only if the restriction of $f$ to each set of three consecutive fibers verifies the property $\mathcal{D}_2^{loc}$. Indeed the neighborhood of a vertex in $M_i$ is included in $M_{i-1} \cup M_i \cup M_{i+1}$ for $i \in \{1, ..., n\}$ (addition mod$n$).

# 4 Resolution method for pseudo-$d$-local $q$-properties of rotagraphs

## 4.1 The auxiliary graph $\vec{\mathcal{G}}(M, \mathcal{P}_q)$:

Given a mixed graph $M = (V, E, A)$ and a pseudo-$d$-local $q$-property $\mathcal{P}_q$ of rotagraphs, we define the directed graph $\vec{\mathcal{G}}(M, \mathcal{P}_q)$, that we will denote by $\vec{\mathcal{G}}$ if there is no ambiguity, as follows:

- the vertices of $\vec{\mathcal{G}}$ are the $q$-labelings $g$ of $\mathcal{P}_q^{loc}(\psi_d(M))$.
- $(u, v) \in A(\vec{\mathcal{G}}) \Leftrightarrow u_{2,d-1} = v_{1,d-1}$ (for all vertices $u$ and $v$ of $\vec{\mathcal{G}}$ not necessarily distinct).

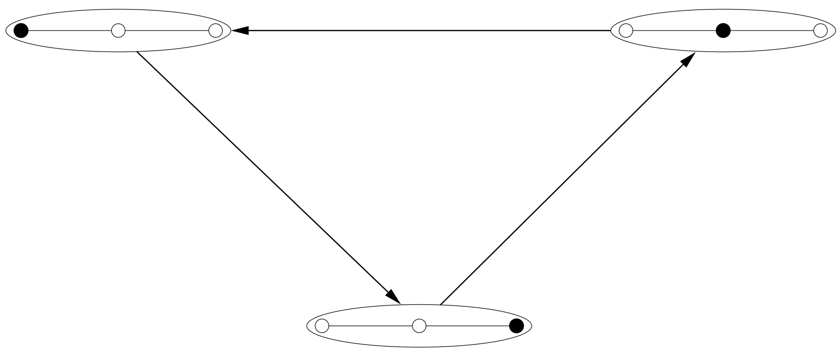

Figure 5: Example of an auxiliary graph $\vec{\mathcal{G}}(M, \mathcal{D}_2)$ for the 2-property $\mathcal{D}_2$ of perfect domination on a mixed graph $M = (V = \{x\}, E = \emptyset, A = \{(x, x)\})$ for rotagraphs.

In the rest of the paper, we will use the same letter to indicate a vertex of $\vec{\mathcal{G}}$ and the $q$-labeling that refers to it. We will then say "the vertex $x$" and the "$q$-labeling $x$".

## 4.2 Circuits of the auxiliary graph

Here, we give a necessary and sufficient condition, based on the existence of directed circuits on the auxiliary graph, for a rotagraph $\omega_n(M)$ to verify a pseudo-$d$-local $q$-property $\mathcal{P}_q$.

**Theorem 1.** *Let $M = (V, E, A)$ be a mixed graph, and $\mathcal{P}_q$ be a pseudo-d-local q-property. For all $n > d$, $\mathcal{P}_q(\omega_n(M)) \neq \emptyset$ if and only if there exists in $\vec{\mathcal{G}}(M, \mathcal{P}_q)$ a directed circuit of length $n$.*

*Proof.*

1. Let us show first that if there exists a directed circuit in $\vec{\mathcal{G}}$ of length $n$, then $\mathcal{P}_q(\omega_n(M)) \neq \emptyset$. So let $\gamma$ be a circuit in $\vec{\mathcal{G}}$: $\gamma = u_1, ..., u_n, u_1$. By construction of $\vec{\mathcal{G}}$, the $q$-labelings $u_1, ..., u_n$ belong to $\mathcal{P}_q^{loc}(\psi_d(M))$. Let $g = u_1 \triangleright_{d-1} ... \triangleright_{d-1} u_{n-d+2}$. So $g \in \mathcal{F}_q(\psi_{n+1}(M))$. From $g$ we define a $q$-labeling $h \in \mathcal{F}_q(\omega_n(M))$ as follows:

   $h_{1,n-1} \leftarrow g_{1,n-1}$ and $h_{n-1,3} \leftarrow g_{n-1,3}$. Since $\mathcal{P}_q$ is a pseudo-$d$-local $q$-property, then the $q$-labeling $h$ belongs to $\mathcal{P}_q(\omega_n(M))$.

2. Then let us show that if $\mathcal{P}_q(\omega_n(M)) \neq \emptyset$ then there exists a directed circuit in $\vec{\mathcal{G}}$ of length $n$. So assume there exists $g \in \mathcal{P}_q(\omega_n(M))$. Since $\mathcal{P}_q$ is a pseudo-$d$-local $q$-property, then $g$ is such that $g_{i,d} \in \mathcal{P}_q^{loc}(\psi_d(M))$, and so $g_{i,d}$ is a vertex of $\vec{\mathcal{G}}$ for all $i \in \{1, ..., n\}$. Moreover $(g_{i,d}, g_{i+1,d}) \in A(\vec{\mathcal{G}})$ for all $i \in \{1, ..., n\}$ . Thus, there exists in $\vec{\mathcal{G}}$ a directed circuit of length $n$: $g = g_{1,d}, ..., g_{n,d}, g_{1,d}$.

□

## 4.3 Circuits detection

In this section, we remind a well-known boolean algebra method that allows to establish the existence of directed circuits of given length in a directed graph $G$ from its adjacency matrix.

We denote by $\mathbb{M}_n(\{0,1\})$ be the set of $n \times n$ binary matrices for $n \in \mathbb{N}$. Given two elements $A, B$ in $\mathbb{M}_n(\{0,1\})$, we define the *product operation* on $A$ and $B$ whose result is a matrix of $\mathbb{M}_n(\{0,1\})$ denoted by $A.B$ such that:

$(A.B)_{ij} = \bigvee_{k=1}^{n}(A_{ik} \wedge B_{kj})$ for $i, j \in \{1, ..., n\}$
where $\vee$ and $\wedge$ are the logical operators "or" and "and".

Notice that the product is associative. Given a matrix $A \in \mathbb{M}_n(\{0,1\})$ the operation $\underbrace{A.A. \; ... \; .A}_{n \; times}$ is denoted by $A^n$.

The following property is well known:

**Property 2.** *Let $\Pi$ be the adjacency matrix of a directed graph $\vec{G}$ and $k$ be any positive integer. For each $i \in \{1, ..., n\}$ we have:*

$(\Pi^k)_{ii} = 1 \Leftrightarrow$ *there exists a directed circuit of length $k$ from $i$ to $i$ in $\vec{G}$.*

The next Lemma, is also well known and used to get constant time algorithms to address problems on fasciagraphs and rotagraphs.

**Lemma 3.** *Let $n \in \mathbb{N}$ and $C \in \mathbb{M}_n(\{0,1\})$. There exists two integers $u$ and $P$ such that, starting from $C^u$, the sequence of powers of $C$ is periodic of period $P$: $\forall k \geq u$, $C^k = C^{u+((k-u) \mod (P))}$.*

*Proof.* There exists $2^{n^2}$ distinct matrices in $\mathbb{M}_{n\times n}(\{0,1\})$. For every integer $k$, $C^k \in \mathbb{M}_{n\times n}(\{0,1\})$, so, in the sequence $C, C^2, ..., C^{2^{n^2+1}}$, there exists two equal elements.

Let $C^T$ be the first duplicated element of this sequence, that is $T$ is such that $C^k \neq C^l$ for all integers $k, l \leq T-1$, and there exists a unique integer $u \leq T-1$ such that $C^u = C^T$.

Let $P = T - u$. Then the sequence of powers of $C$ is

$$C, C^2, ..., C^{u-1}, [C^u, C^{T-1}], [C^u, C^{T-1}], ...$$

where $[C^u, C^{T-1}]$ is the sequence of the $P$ consecutive powers of $C$ from $u$ to $T-1$. So, for every integer $k \geq T$,

$$C^k = C^{u+((k-u) \mod (P))}$$

□

## 4.4 Presentation of the algorithm

In this section, we propose an algorithm that, given a mixed graph $M = (V, E, A)$, a pseudo-$d$-local $q$-property $\mathcal{P}_q$ and an integer $n$, decides if $\omega_n(M)$ satisfies $\mathcal{P}_q$. In order to make this algorithm feasible, the following condition is essential:

**Checkability condition:** A pseudo-$d$-local $q$-property is called "*checkable*" if there exists a procedure which decides in finite time if an element $f$ of $\mathcal{F}_q(\psi_d(M))$ belongs to $\mathcal{P}_q^{loc}(\psi_d(M))$ for any fixed mixed graph $M$.

**Remark 5.** *In all of our previous examples, the $q$-properties are checkable.*

The algorithm is made of two parts: given a mixed graph $M$ and a pseudo-$d$-local $q$-property $\mathcal{P}_q$, a preprocess that is executed only once computes the closed formula, and another part which uses the closed formula to decide, for any given $n$, if $\omega_n(M)$ satisfies $\mathcal{P}_q$.

**Theorem 4.** *Given $M = (V, E, A)$ a mixed graph and $\mathcal{P}_q$ a checkable pseudo-$d$-local $q$-property, Algorithm 1 gives, in an execution time independent of $n$, the formula $\tau$ that allows Algorithm 2 to decide if $\mathcal{P}_q(\omega_n(M)) \neq \emptyset$ for all integers $n$ : $\mathcal{P}_q(\omega_n(M)) \neq \emptyset$ if and only if $\tau(n) = 1$.*

**Algorithm 1**: Preprocessing procedure for a pseudo-$d$-local $q$-property $\mathcal{P}_q$ and a mixed graph $M = (V, E, A)$.

**Data**: A mixed graph $M = (V, E, A)$, a checkable pseudo-$d$-local $q$-property $\mathcal{P}_q$

**Result**: Integers $u$ and $P$ of Lemma 3, and a binary vector $\mathcal{T}$ of length $T-1$ such that for all integers $i < T$ $\begin{cases} \mathcal{T}_i = 1 \text{ if and only if there exists } j \leq |V(\vec{\mathcal{G}})| \text{ such that } (\Pi^i)_{jj} = 1 \\ \mathcal{T}_i = 0 \text{ otherwise} \end{cases}$

**begin**
- **Computation of the auxiliary graph $\vec{\mathcal{G}} = (\vec{V}, \vec{A})$:**
- **Computation of the vertex set $\vec{V}$:**
- $\vec{V} \leftarrow \emptyset$ ;
- Generate all the elements of $\mathcal{F}_q(\psi_d(M))$;
- **forall** $f \in \mathcal{F}_q(\psi_d(M))$ **do**
  - Check if $f \in \mathcal{P}_q^{loc}(\psi_d(M))$;
  - **if** $f \in \mathcal{P}_q^{loc}(\psi_d(M))$ **then**
    - $V \leftarrow \{f\} \cup V$ ;
- **Computation of the set of arcs $\vec{A}$ and adjacency matrix $\Pi$ of $\vec{\mathcal{G}}$:**
- $\vec{A} \leftarrow \emptyset$;
- $\Pi_{uv} \leftarrow 0$ for all $(u, v) \in \vec{V}^2$ ;
- **forall** $(u, v) \in \vec{V}^2$ **do**
  - **if** $u_{2,d-1} = v_{1,d-1}$ **then**
    - $\vec{A} \leftarrow \{(u, v)\} \cup \vec{A}$ ;
    - $\Pi_{uv} \leftarrow 1$;
- **Computation of powers of $\Pi$:**
- $i \leftarrow 1$;
- $T \leftarrow 0$;
- **while** $T = 0$ **do**
  - $j \leftarrow 1$;
  - **while** $\Pi^i \neq \Pi^j$ **do**
    - $j \leftarrow j + 1$;
  - **if** $j = i$ **then**
    - $i \leftarrow i + 1$;
    - **Computation of the new matrix $\Pi^i$:**
    - $\Pi^i \leftarrow \Pi^{i-1}.\Pi$;
  - **else**
    - $T \leftarrow i$;
    - $u \leftarrow j$;
    - $P \leftarrow T - u$;
- **Computation of $\mathcal{T}$:**
- $\mathcal{T} \leftarrow 0$;
- **forall** $i \in \{1, ..., T-1\}$ **do**
  - **forall** $j \in \{1, ..., |V(\vec{\mathcal{G}})|\}$ **do**
    - $\mathcal{T}_i \leftarrow (\Pi^i)_{jj} \vee \mathcal{T}_i$;

**end**

**Algorithm 2**: An answer to the question: Does there exist a $q$-labeling $f \in \mathcal{P}_q(\omega_n(M))$?

**Data**: An integer $n$, the vector $\mathcal{T}$, integers $u$ and $P$ obtained by Algorithm 1
**Result**: Yes if there exists $f \in \mathcal{P}_q(\omega_n(M))$, No otherwise

```
begin
    if n < u then
        k ← n;
    else
        k ← u + ((n − u) mod (P));
    if T_k = 1 then
        Answer Yes;
    else
        Answer No;
end
```

*Proof.* By Theorem 1, we know that $\mathcal{P}(\omega_n(M)) \neq \emptyset$ if and only if there exists a directed circuit of length $n$ in the directed graph $\vec{\mathcal{G}}$, or in other words, if and only if there exists $i \leq |V(\vec{\mathcal{G}})|$ such that $(\Pi^n)_{ii} = 1$. Thanks to Lemma 3 we know that for $n \geq u$, there exists in $\vec{\mathcal{G}}$ a directed circuit of length $n$ if and only if there exists one of length $u+((n-u) \bmod (P))$. For all $n < P+u$, there exists a directed circuit of length $n$ in $\vec{\mathcal{G}}$ if and only if $\mathcal{T}[u+((n-u) \mod (P))] = 1$. Let $\tau$ be the following closed formula: $\tau(n) = 1 \Leftrightarrow \mathcal{T}[u+((n-u) \mod (P))] = 1$. Then $\tau$ is such that for all $n$, $\tau(n) = 1$ if and only if $\mathcal{P}(\omega_n(M)) \neq \emptyset$.

The execution time of Algorithm 1 is independent of $n$, and depends only on the mixed graph $M = (V, E, A)$ and on the property $\mathcal{P}_q$. Since the mixed graph and the property $\mathcal{P}_q$ are fixed, Algorithm 1 is executed only once. Then, the output of this algorithm (integers $u$ and $P$ and the vector $\mathcal{T}$) is fixed. It won't ever change whatever the value of $n$. So Algorithm 1 gives, in a constant time (independent of $n$), the formula that allows Algorithm 2 to decide if $\mathcal{P}(\omega_n(M)) \neq \emptyset$ or not.

□

# 5 Decision problems and $q$-properties of fasciagraphs

In this section we present, for fasciagraphs, results similar to those developed for rotagraphs.

The notion of pseudo-$d$-locality for the case of $q$-properties of fasciagraphs is slightly more complex than in the context of $q$-properties of rotagraphs since it is necessary to deal with "side effects". We are redefining it as follows:

**Definition 21** (Pseudo-$d$-locality for fasciagraphs)**.** *For an integer $d \geq 2$, a $q$-property $\mathcal{P}_q$ of fasciagraphs is said to be pseudo-$d$-local if there exists three $q$-properties $\mathcal{P}_q^1$, $\mathcal{P}_q^2$ and $\mathcal{P}_q^3$ on fasciagraphs of length $d$ such that for every mixed graph $M = (V, E, A)$, for every integer $n > d$, and for every $q$-labeling*

$f \in \mathcal{F}_q(\psi_n(M))$,

$$f \in \mathcal{P}_q(\psi_n(M)) \Leftrightarrow \begin{cases} f_{1,d} \in \mathcal{P}_q^1(\psi_d(M)) \\ f_{i,d} \in \mathcal{P}_q^2(\psi_d(M)) \text{ for every } i \in \{2, ..., n-d\} \\ f_{n-d+1,d} \in \mathcal{P}_q^3(\psi_d(M)) \end{cases}$$

Here, the $q$-property $\mathcal{P}_q^2$ plays the same role as the $q$-property $\mathcal{P}_q^{loc}$ presented in the definition of pseudo-$d$-locality for the case of rotagraphs, $\mathcal{P}_q^1$ and $\mathcal{P}_q^3$ being defined only to deal with side effects.

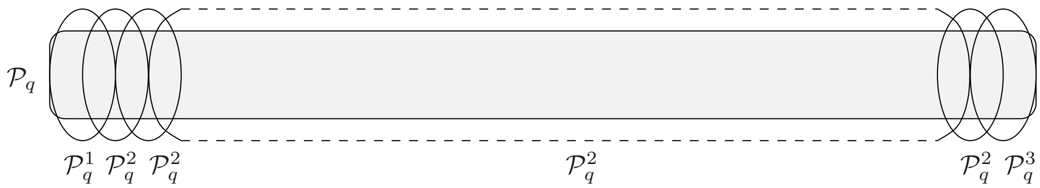


Figure 6: Representation of a pseudo-$d$-local $q$-property $\mathcal{P}_q$.

**Example** Let us consider again the 2-property $\mathcal{D}_2$ of perfect domination (see Section 3). The 2-property $\mathcal{D}_2$ was pseudo-3-local for rotagraphs.

$\mathcal{D}_2$ is also a pseudo-3-local 2-property of fasciagraphs. Indeed, for any mixed graph $M = (V, E, A)$ and 2-labeling $f \in \mathcal{F}_2(\psi_3(M))$, we define $\mathcal{D}_2^1$, $\mathcal{D}_2^2$ and $\mathcal{D}_2^3$ the following 2-properties of fasciagraphs on 3 fibers:

- $f \in \mathcal{D}_2^1(\psi_3(M)) \Leftrightarrow \forall u \in V(M_1) \cup V(M_2)$, $u$ is uniquely-dominated by $f^{-1}(1)$,
- $f \in \mathcal{D}_2^2(\psi_3(M)) \Leftrightarrow \forall u \in V(M_2)$, $u$ is uniquely-dominated by $f^{-1}(1)$,
- $f \in \mathcal{D}_2^3(\psi_3(M)) \Leftrightarrow \forall u \in V(M_2) \cup V(M_3)$, $u$ is uniquely-dominated by $f^{-1}(1)$,

It is easy to see that a 2-labeling $f$ of a fasciagraph $\psi_n(M)$ corresponds to a perfect dominating set if and only if

- the restriction of $f$ induced by the three first consecutive fibers of $\psi_n(M)$ satisfy $\mathcal{D}_2^1$,
- the restriction of $f$ induced by three consecutive fibers of $\psi_n(M)$ from fiber 2 to $n-3$ satisfy $\mathcal{D}_2^2$,
- the restriction of $f$ induced by the three last consecutive fibers of $\psi_n(M)$ satisfy $\mathcal{D}_2^3$.

# 6 Resolution method for pseudo-$d$-local $q$-properties of fasciagraphs

## 6.1 The auxiliary graph $\vec{\mathcal{G}}(M, \mathcal{P}_q)$:

The construction of the directed graph $\vec{\mathcal{G}}(M, \mathcal{P}_q)$ for pseudo-$d$-local $q$-properties of fasciagraphs is similar to the one for rotagraphs.

Given a mixed graph $M = (V, E, A)$ and a pseudo-$d$-local $q$-property $\mathcal{P}_q$ of fasciagraphs, we define the directed graph $\vec{\mathcal{G}} = \vec{\mathcal{G}}(M, \mathcal{P}_q)$, that we will denote by $\vec{\mathcal{G}}$ if there is no ambiguity, as follows:

- the vertices of $\vec{\mathcal{G}}$ are the $q$-labelings in $\mathcal{P}_q^1(\psi_d(M))$, $\mathcal{P}_q^2(\psi_d(M))$ and $\mathcal{P}_q^3(\psi_d(M))$ and two special vertices $s$ and $t$:

$$V(\vec{\mathcal{G}}) = \mathcal{P}_q^1(\psi_d(M)) \cup \mathcal{P}_q^2(\psi_d(M)) \cup \mathcal{P}_q^3(\psi_d(M)) \cup \{s, t\}$$

- for any $q$-labelings $u$ and $v$ not necessarily distinct:
  $(u, v) \in A(\vec{\mathcal{G}}) \Leftrightarrow u \in \mathcal{P}_q^i(\psi_d(M)),\ v \in \mathcal{P}_q^j(\psi_d(M))$ and $u_{2,d-1} = v_{1,d-1}$ with $i \in \{1, 2\}$ and $j \in \{2, 3\}$,
  $(s, u) \in A(\vec{\mathcal{G}}) \Leftrightarrow u \in \mathcal{P}_q^1(\psi_d(M))$,
  $(u, t) \in A(\vec{\mathcal{G}}) \Leftrightarrow u \in \mathcal{P}_q^3(\psi_d(M))$.

**Remark 6.** *Notice that a same vertex can satisfy several properties among $\mathcal{P}_q^1$, $\mathcal{P}_q^2$ and $\mathcal{P}_q^3$ (see Figure 7).*

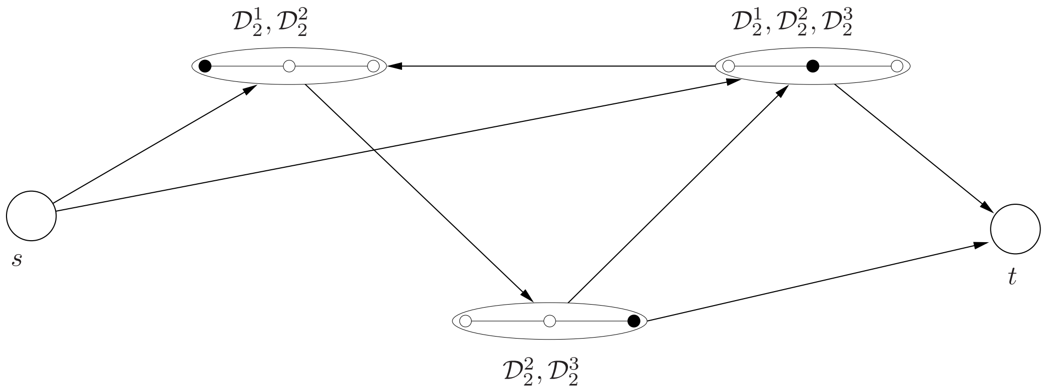


Figure 7: Example of the auxiliary graph $\vec{\mathcal{G}}(M, \mathcal{D}_2)$ for the 2-property $\mathcal{D}_2$ of perfect domination for fasciagraphs on a mixed graph $M = (V = \{x\}, E = \emptyset, A = \{(x, x)\})$. The properties satisfied by the labelings are indicated.

## 6.2 Directed paths of the auxiliary graph

Here, we give a necessary and sufficient condition, based on the existence of directed paths on the auxiliary graph, for a fasciagraph $\psi_n(M)$ to verify a pseudo-$d$-local $q$-property $\mathcal{P}_q$.

**Theorem 5.** *Let $M = (V, E, A)$ be a mixed graph and $\mathcal{P}_q$ be a pseudo-$d$-local $q$-property. For all $n > d$, $\mathcal{P}_q(\psi_n(M)) \neq \emptyset$ if and only if there exists in $\vec{\mathcal{G}}(M, \mathcal{P}_q)$ a directed path from $s$ to $t$ of length $n - d + 2$.*

*Proof.* We show that there is a 1-to-1 correspondence between labelings in $\mathcal{P}_q(\psi_n(M))$ and some paths of $\vec{\mathcal{G}}$.

1. Let us show first that if there exists a directed path $\gamma = s, u_1, ..., u_{n-d+1}, t$ from $s$ to $t$ in $\vec{\mathcal{G}}$ of length $n-d+2$, then $\mathcal{P}_q(\psi_n(M)) \neq \emptyset$. By construction of $\vec{\mathcal{G}}$, $u_1 \in \mathcal{P}_q^1(\psi_d(M))$, $u_2, ..., u_{n-d} \in \mathcal{P}_q^2(\psi_d(M))$, and $u_{n-d+1} \in \mathcal{P}_q^3(\psi_d(M))$. Let $g = u_1 \triangleright ... \triangleright u_{n-d+1}$. Since $\mathcal{P}_q$ is a pseudo-$d$-local $q$-property, then the $q$-labeling $g$ belongs to $\mathcal{P}(\psi_n(M))$.

2. Then let us show that if $\mathcal{P}_q(\psi_n(M)) \neq \emptyset$ then there exists a directed path from $s$ to $t$ in $\vec{\mathcal{G}}$ of length $n-d+2$. So assume there is $g \in \mathcal{P}_q(\psi_n(M))$. Since $\mathcal{P}_q$ is a pseudo-$d$-local $q$-property, then $g$ is such that $g_{1,d} \in \mathcal{P}_q^1(\psi_d(M))$, $g_{i,d} \in \mathcal{P}_q^2(\psi_d(M))$ for all $i \in \{2, ..., n-d\}$, and $g_{n-d+1,d} \in \mathcal{P}_q^3(\psi_d(M))$. By definition of $\vec{\mathcal{G}}$, $g_{i,d}$ is a vertex of $\vec{\mathcal{G}}$ for all $i \in \{1, ..., n-d+1\}$, and $(s, g_{1,d})$ and $(g_{n-d+1,d}, t)$ are arcs of $\vec{\mathcal{G}}$. Moreover, for all $i \in \{1, ..., n-d\}$, $(g_{i,d}, g_{i+1,d}) \in A(\vec{\mathcal{G}})$. Thus, there exists in $\vec{\mathcal{G}}$ a directed path from $s$ to $t$ of length $n-d+2$.

□

### 6.3 Paths detection

The algorithm is based on the same method of boolean algebra used for the $q$-properties of rotagraphs. Let $\Pi$ be the adjacency matrix of the directed graph $\vec{\mathcal{G}}$ and $k$ be any positive integer, there exists a directed path in $\vec{\mathcal{G}}$ of length $k$ from $i$ to $j$ if and only if $(\Pi^k)_{ij} = 1$.

In the same way as before, it only remains to notice that there exists a period in the sequence of powers of $\Pi$. Moreover we need to keep only one row vector up to date rather than the entire matrix. Indeed, we just need to know if there is a path in $\vec{\mathcal{G}}$ from $s$ to $t$:

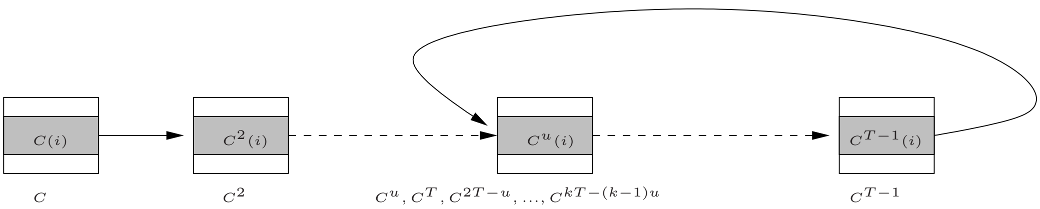


Figure 8: Visualization of the pseudo-period of powers of matrices.

**Definition 22.** *Given an integer $n$, a matrix $C \in \mathcal{M}(n \times n)$ and an integer $i \leq n$, we denote by $C(i)$ the $i$-th row vector of $C$.*

**Lemma 6.** *Let $n \in \mathbb{N}$, $C \in \mathbb{M}_n(\{0,1\})$ and $i \leq n$. There exists two integers $u$ and $P$ such that for all $k \geq u$, $C^k(i) = C^{u+((k-u) \mod (P))}(i)$.*

*Proof.* There exists $2^n$ distinct binary vectors of size $n$ so, given an integer $i$, there exists two equal elements in the sequence $C(i), C^2(i), ..., C^{2^n+1}(i)$.

Let $C^T(i)$ the first duplicated element of this sequence, that is $T$ is such that $C^k(i) \neq C^l(i)$ for all integers $k, l \leq T-1$, and there exits a unique integer $u \leq T-1$ such that $C^u(i) = C^T(i)$.

Let $P = T-u$. Then the sequence of $i$-th rows of powers of $C$ is $C(i)$,$C^2(i)$,..., $C^{u-1}(i)$,$[C^u(i), C^{T-1}(i)]$,$[C^u(i), C^{T-1}(i)]$,...

where $[C^u(i), C^{T-1}(i)]$ is the sequence of the $i$-th rows of the $P$ consecutive powers of $C$ from $u$ to $T-1$. So, for every integer $k \geq T$,

$$C^k(i) = C^{u+((k-u) \bmod (P))}(i)$$

□

## 6.4 Presentation of the algorithm

In this section, we propose an algorithm that, given a mixed graph $M = (V, E, A)$, a pseudo-$d$-local $q$-property $\mathcal{P}_q$ and an integer $n$, decides if $\psi_n(M)$ satisfies $\mathcal{P}_q$. In order to make this algorithm feasible, the following condition is required:

**Checkability condition:**A pseudo-$d$-local $q$-property of fasciagraphs, is called "*checkable*" if there exists a procedure which decides in finite time if, a given $q$-labeling of a fasciagraph of size $d$ satisfies $\mathcal{P}_q^1$, $\mathcal{P}_q^2$ or $\mathcal{P}_q^3$.

The algorithm is made of two parts: a preprocessing that is executed only once, and another part corresponding to the response algorithm using results of the preprocess. This algorithm is similar to the one presented before for rotagraphs.

**Theorem 7.** *Given $M = (V, E, A)$ a mixed graph and $\mathcal{P}_q$ a checkable pseudo-$d$-local $q$-property, Algorithm 3 gives, in an execution time independent of $n$, the closed formula $\varphi$ that allows Algorithm 4 to decide if $\mathcal{P}_q(\psi_n(M)) \neq \emptyset$ for all integers $n$ : $\mathcal{P}_q(\psi_n(M)) \neq \emptyset$ if and only if $\varphi(n) = 1$.*

*Proof.* By Theorem 5, we know that $\mathcal{P}(\psi_n(M)) \neq \emptyset$ if and only if there exists a directed path from $s$ to $t$ of length $n-d+2$ in the directed graph $\vec{\mathcal{G}}$, or in other words, if and only if $\Pi^{n-d+2}(s)_t = 1$. Thanks to Lemma 6 we know that for $n \geq u$, there exists in $\vec{\mathcal{G}}$ a directed path from $s$ to $t$ of length $n-d+2$ if and only if there exists one of length $u + ((n-d+2-u) \bmod (P))$. However for all $n-d+2 < P+u$, there exists a directed path of length $n-d+2$ in $\vec{\mathcal{G}}$ if and only if $\mathcal{T}[u + ((n-d+2-u) \mod (P))] = 1$. Let $\varphi$ be the following closed formula: $\varphi(n) = 1 \Leftrightarrow \mathcal{T}[u + ((n-u) \mod (P))] = 1$. Then $\varphi$ is such that for all $n$, $\varphi(n) = 1$ if and only if $\mathcal{P}(\psi_n(M)) \neq \emptyset$.

The execution time of Algorithm 3 is independent of $n$, and depends only on the mixed graph $M = (V, E, A)$ and on the property $\mathcal{P}_q$. Since the mixed graph and the property $\mathcal{P}_q$ are fixed, Algorithm 3 is executed only once. Then, results of this algorithm (integers $u$ and $P$ and the vector $\mathcal{T}$) are as well fixed. They won't ever change whatever the value of $n$. So algorithm 3 gives, in a constant time (independent of $n$), the formula that allows Algorithm 4 to decide if $\mathcal{P}(\psi_n(M)) \neq \emptyset$ or not.

□

**Algorithm 3**: Preprocessing procedure for a pseudo-$d$-local $q$-property $\mathcal{P}_q$ and a mixed graph $M = (V, E, A)$.

**Data**: A mixed graph $M = (V, E, A)$, a checkable pseudo-$d$-local $q$-property $\mathcal{P}_q$ satisfying the verifying condition

**Result**: Integers $u$ and $P$ of Lemma 6, and a vector $\mathcal{T}$ of size $T-1$ valued in $\{0,1\}$ such that for all integers $i < T \begin{cases} \mathcal{T}_i = 1 \text{ if and only if } \Pi^i(s)_t = 1 \\ \mathcal{T}_i = 0 \text{ otherwise} \end{cases}$

```
begin
    Computation of the auxiliary graph G⃗ = (V⃗, A⃗):
    Computation of the vertices set V⃗:
    V ← {s, t} ;
    Generate all the elements of F_q(ψ_d(M));
    forall f ∈ F_q(ψ_d(M)) do
        Check if f ∈ P_q^1(ψ_d(M)) or f ∈ P_q^2(ψ_d(M)) or f ∈ P_q^3(ψ_d(M));
        if f ∈ P_q^1(ψ_d(M)) ∨ f ∈ P_q^2(ψ_d(M)) ∨ f ∈ P_q^3(ψ_d(M)) then
            V ← {f} ∪ V ;
    Computation of the set of arcs A⃗ and adjacency matrix Π of G⃗:
    A⃗ ← ∅;
    Π_uv ← 0 for all (u, v) ∈ V⃗^2 ;
    forall (u, v) ∈ V⃗ ∩ P_q^k(ψ_d(M)) × V⃗ ∩ P_q^l(ψ_d(M)) for k ∈ {1, 2} and l ∈ {2, 3} do
        if u_{2,d−1} = v_{1,d−1} then
            A⃗ ← {(u, v)} ∪ A⃗ ;
            Π_uv ← 1;
    forall u ∈ P_q^1(ψ_d(M)) do
        A⃗ ← {(s, u)} ∪ A⃗ ;
        Π_su ← 1;
    forall u ∈ P_q^3(ψ_d(M)) do
        A⃗ ← {(u, t)} ∪ A⃗ ;
        Π_ut ← 1;
    Computation of the vectors of the s-th row of powers of Π:
    i ← 1;
    T ← 0;
    while T = 0 do
        j ← 1;
        while Π^i(s) ≠ Π^j(s) do
            j ← j + 1;
        if j = i then
            i ← i + 1;
            Computation of the new vector Π^i(s):
            Π^i(s) ← Π^{i−1}(s).Π;
        else
            T ← i;
            u ← j;
            P ← T − u;
    Computation of T:
    T ← 0;
    forall i ∈ {1, ..., T − 1} do
        T_i ← Π^i(s)_t;
end
```

**Algorithm 4**: An answer to the question: Does there exist a $q$-labeling $f \in \mathcal{P}_q(\psi_n(M))$?

**Data**: An integer $n$, the vector $\mathcal{T}$, integers $u$ and $P$ obtained by Algorithm 3
**Result**: Yes if there exists $f \in \mathcal{P}_q(\psi_n(M))$, No otherwise

```
begin
    if n < u then
        k ← n;
    else
        k ← u + ((n − u) mod (P));
    if T_k = 1 then
        Answer Yes;
    else
        Answer No;
end
```

# 7 Conclusion

In this paper, we gave a theoretical framework that allowed us to to unify and expand, under a generic approach, results already known concerning solving methods of several combinatorial problems for the class of rotagraphs and fasciagraphs (see [1],[3],[8],[9],[10],[12]). In the first section, we have given all definitions needed to the comprehension of our approach. We have introduced pseudo-$d$-locality from the point of view of $q$-properties of rotagraphs. Then we have shown that for all checkable pseudo-$d$-local $q$-properties, there exists a method which, given a fixed fiber $M$, gives in constant time, a closed formula allowing to decide, for every integer $n$, if the rotagraph $\omega_n(M)$ satisfies these properties. Finally, we showed similar for the case of $q$-properties of fasciagraphs.

In a future paper, we will adapt our method for optimization problems and implement it to some cases with small size of fiber, as it has been done in [8],[12] and other papers. We will furthermore propose an alternative approach using Monadic Second Order Logic in order to propose other algorithms and give another characterization of pseudo-$d$-local $q$-properties.

# A

We present a counter-example to a part of one of the theorems presented by Klavžar and Vesel in [8]. We begin by giving definitions of heredity and $d$-locality as presented in [8] but with our vocabulary.

**Definition 23.** *A $q$-property of rotagraphs $\mathcal{P}_q$ is said to be hereditary if, for every mixed graph $M = (V, E, A)$, for every integer $n$, and for every $q$-labeling* *of vertices* $f \in \mathcal{F}_q(\omega_n(M))$[4]*:*

$$f \in \mathcal{P}_q(\omega_n(M)) \Rightarrow f_{i,k} \in \mathcal{P}_q(\psi_k(M)) \; \forall i,k \in \{1,...,n\}$$

**Definition 24.** *Let $d \geq 2$ be an integer. A $q$-property $\mathcal{P}_q$ is said to be $d$-local, if for any rotagraph $\omega_n(M)$ and for any $q$-labeling $f \in \mathcal{F}_q(\omega_n(M))$*[5]*,*

$$\forall i \in \{1,...,n\}, f_{i,d} \in \mathcal{P}_q(\psi_d(M)) \Rightarrow f \in \mathcal{P}_q(\omega_n(M))$$

In [8], notions of heredity and of $d$-locality of $q$-properties for the case of $q$-properties of fasciagraphs are also defined.

**Proposition 8.** *The $q$-properties of rotagraphs which are both hereditary and $d$-local, are also pseudo-$d$-local $q$-properties of rotagraphs.*

*Proof.* Let $\mathcal{P}_q$ be a hereditary and $d$-local $q$-property of rotagraphs. By Definition 20, $\mathcal{P}_q$ is also pseudo-$d$-local with $\mathcal{P}_q^{loc} = \mathcal{P}_q$.

□

**Proposition 9.** *The $q$-properties of fasciagraphs which are both hereditary and $d$-local are also pseudo-$d$-local $q$-properties of fasciagraphs.*

*Proof.* Let $\mathcal{P}_q$ be a hereditary and $d$-local $q$-property of fasciagraphs. By Definition 20, $\mathcal{P}_q$ is also pseudo-$d$-local with $\mathcal{P}_q^1 = \mathcal{P}_q^2 = \mathcal{P}_q^3 = \mathcal{P}_q$.

□

**The directed graph $D_d(M, \mathcal{P}_q)$** Given $\mathcal{P}_q$ a hereditary and $d$-local $q$-property of rotagraphs, and $\omega_n(M)$ a rotagraph ($n \geq d+1$), the directed graph $D_d(M, \mathcal{P}_q)$ is defined as follows: vertices are $q$-labelings of $\mathcal{P}(\psi_2(M))$, arcs are defined according to the value of $d$:

- if $d = 2$: for every $f, g \in V(D_d(M, \mathcal{P}_q))$, $(f, g) \in A(D_d(M, \mathcal{P}_q))$ if and only if $f_{2,1} = g_{1,1}$,
- if $d = 3$: for every $f, g \in V(D_d(M, \mathcal{P}_q))$, $(f, g) \in A(D_d(M, \mathcal{P}_q))$ if and only if $f_{2,1} = g_{1,1}$ and $f \triangleright g \in \mathcal{P}_q(\psi_3(M))$
- if $d \geq 4$, the graph has two types of arcs:
  - classical arcs: for every $f, g \in V(D_d(M, \mathcal{P}_q))$, $(f, g) \in A(D_d(M, \mathcal{P}_q))$ if and only if $f_{2,1} = g_{1,1}$
  - $d$-arcs: for every $f, g \in V(D_d(M, \mathcal{P}_q))$, $(f, g)$ is a $d$-arc if and only if there exists a path $\gamma = f_1, f_2, ..., f_{d-1}$ of length $d-2$ whose the $q$-labeling $g = f_1 \triangleright_1 f_2 \triangleright_1, ..., \triangleright_1 f_{d-1}$ is such that $g \in \mathcal{P}_q(\psi_d(M))$,

We present below, with our vocabulary, the assertion presented in [8] under the denomination "Theorem 1":

[4] In [8], $q$-labelings are only labeling functions of the vertices. In our approach, the definition domain of $q$-labelings includes vertices and edges letting thus address a wider range of problems.

[5] The definition given here has been modified for convenience in comparison to that given in [8]. Thus, our definition of $d$-locality ultimately corresponds to the $d+1$-locality as described in [8].

**Assertion 10.** *Let $\mathcal{P}_q$ be a hereditary and d-local q-property, and let $\omega_n(M)$ be a rotagraph ($n \geq d+1 \geq 3$). The set $\mathcal{P}_q(\omega_n(M)) \neq \emptyset$ if and only if $D_d(M, \mathcal{P}_q)$ contains vertices $f_1, f_2, ..., f_n$ (not necessarily distinct) such that, for all $i \in \{1, ..., n\}$:*

- *$(f_i, f_{i+1})$ is an arc of $D_d(M, \mathcal{P}_q)$*
- *if $d \geq 4$, $(f_i, f_{i+d-2})$ is a d-arc of $D_d(M, \mathcal{P}_q)$*

For $d \in \{2, 3\}$, the Assertion 10 is correct as proved in [8]. It can also be easily deduced from our previous results (Propositions 8 and 9 and Theorems 1 and 5).

For $d \geq 4$ the proof of Assertion 10 is not correct. We present a counter-example for $d = 5$ which can be generalized for any $d \geq 5$. For that purpose, we define the 2-property $\mathcal{C}ent_2$ of (rota)graphs as follows:

**Definition 25.** *Let $G = (V, E)$ be an undirected graph. A chain in $G$ is a path where no vertex appears twice.*

A (rota)graph $R$ verifies $\mathcal{C}ent_2$ if and only if there exists a 2-labeling $f$ of $R$ such that for any chain $v, w, x, y, z$, we have $f(v) = f(w) = f(y) = f(z) = 1$ and $f(x) = 2$ (see Figure 9).

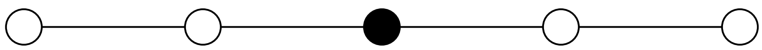

Figure 9: A chain on 5 vertices satisfies $\mathcal{C}ent_2$: the full vertex is labeled 2, empty vertices are labeled 1

This 2-property is hereditary and 5-local. Indeed, if $R$ satisfies $\mathcal{C}ent_2$, then every subgraph of $R$ satisfies $\mathcal{C}ent_2$, and if all 5-tuples of consecutive fibers of $R$ satisfy $\mathcal{C}ent_2$ then $R$ satisfies $\mathcal{C}_2$ since every chain on 5 vertices of $R$ is contained in a 5-tuple of consecutive fibers of $R$.

Moreover, it is easy to check that, a graph that contains a chain on 6 vertices, can't satisfy the property $\mathcal{C}ent_2$.

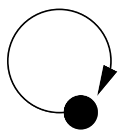

Figure 10: A mixed graph $M$

For the mixed graph $M = (V, E, A)$ of Figure 10, the rotagraph $\omega_n(M)$ is a cycle of length $n$.

Let us build the directed graph $D_5(M, \mathcal{C}ent_2)$ (see Figure 11) :

The arc $(Z, Z)$ (dotted in figure) is a $d$-arc. Indeed the 2-labeling $g = Z \triangleright_1 Y \triangleright_1 W \triangleright_1 Z$ is such that $g \in \mathcal{C}ent_2(\psi_5(M))$.

The function $g = Z \triangleright_1 Z \triangleright_1 ... \triangleright_1 Z$ satisfies both conditions of the assertion, and yet, for every $n \geq 6$, $\omega_n(M)$ does not satisfy $\mathcal{C}ent_2$, since it contains a chain of 6 vertices.

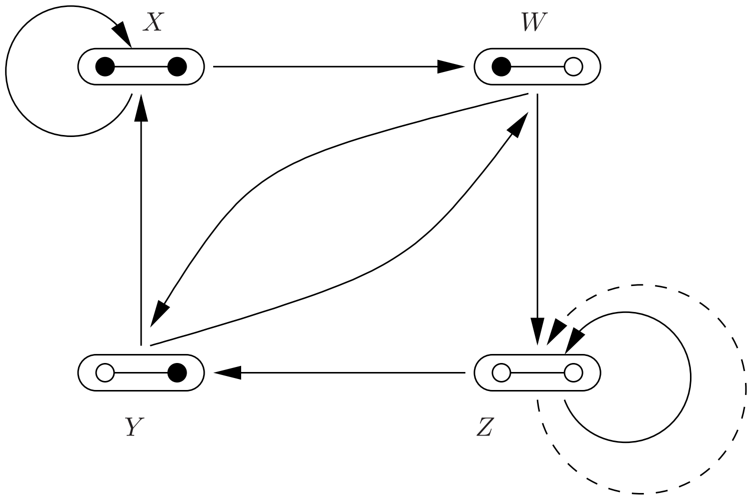


Figure 11: The graph $D_5(M, \mathcal{C}ent_2)$

# References

[1] Marc Daniel, Sylvain Gravier, and Julien Moncel. Identifying codes in some subgraphs of the square lattice. *Theoritical Computer Science*, 319(1-3):411–421, 2004.

[2] Walter Günther. The automorphism group of roto- and fasciagraphs. *Communication in Mathematical and Computer Chemistry*, 14:3–42, 1983.

[3] Sylvain Gravier, Julien Moncel, and Charles Payan. A generalization of the pentomino exclusion problem: Dislocation of graphs. *Discrete Mathematics*, 307(3-5):435–444, 2007.

[4] Pranava K. Jha, Sandi Klavzar, and Aleksander Vesel. L(2,1)-labeling of direct product of paths and cycles. *Discrete Applied Mathematics*, 145(2):317–325, 2005.

[5] Martin Juvan, Bojan Mohar, Ante Graovac, Sandi Klavžar, and Janez Žerovnik. Fast computation of the Wiener index of fasciagraphs and rotagraphs. *Journal of Chemical Information and Computer Sciences*, 35(5):834–840, 1995.

[6] Martin Juvan, Bojan Mohar, and Janez Zerovnik. Distance-related invariants on polygraphs. *Discrete Applied Mathematics*, 80(1):57–71, 1997.

[7] Martin Juvan, Marko Petkovšek, Ante Graovac, Aleksander Vesel, and Janez Žerovnik. The szeged index of fasciagraphs. *Communication in Mathematical and Computer Chemistry*, 49:47–66, 2003.

[8] Sandi Klavžar and Aleksander Vesel. Computing graph invariants on rotagraphs using dynamic algorithm approach: the case of (2,1)-colorings and independence numbers. *Discrete Applied Mathematics*, 129(2-3):449–460, 2003.

[9] Sandi Klavžar and Janez Žerovnik. Algebraic approach to fasciagraphs and rotagraphs. *Discrete Applied Mathematics*, 68(1-2):93–100, 1996.

[10] Marilynn Livingston and Quentin F. Stout. Constant time computation of minimum dominating sets. *Congressus Numerantium*, 105:116–128, 1994.

[11] RH Möhring. Graph problems related to gate matrix layout and PLA folding. *Computing Supplementum*, 7:17–51, 1990.

[12] Janez Žerovnik. Deriving formulas for domination numbers of fasciagraphs and rotagraphs. *FCT '99: Proceedings of the 12th International Symposium on Fundamentals of Computation Theory : Springer. Lecture Notes in Computer Science*, 1684:559–568, 1999.

[13] Janez Žerovnik. Deriving formulas for the pentomino exclusion problem. *Preprint Series*, 42, 2004.

[14] Douglas M. Van Wieren. Critical cyclic patterns related to the domination number of the torus. *Discrete Mathematics*, 307(3-5):615–632, 2007.

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