Les cahiers Leibniz

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On 3-colorable P_5-free graphs

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Abstract

We consider the class of P_5-free 3-colorable graphs. We give a complete description of the structure of the graphs in that class. We derive a linear time algorithm that tests membership in the class. The algorithm also finds a maximum weight stable set in linear time.

Keywords: graph algorithm, structure theorem, stability

1 Introduction

In a graph G, a *stable set* (also called *independent* set) is any subset of pairwise non-adjacent vertices. The MAXIMUM STABLE SET PROBLEM (henceforth MSS) is the problem of finding a stable set of maximum size. In the weighted version of this problem, each vertex v of G has a weight $f(v)$, and the weight of any subset of vertices is defined as the total weight of its elements. The MAXIMUM WEIGHTED STABLE SET PROBLEM (MWSS) is then the problem of finding a stable set of maximum weight. MSS (and consequently MWSS) is NP-hard in general, even under strong restrictions [4].

Given a fixed graph F, a graph G *contains* F when F is isomorphic to an induced subgraph of G. A graph G is said to be F*-free* if it does not contain F. Let us say here that F is *special* if every component of F is a tree with no vertex

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of degree at least four and with at most one vertex of degree three. Alekseev [1] proved that MSS remains NP-complete in the class of F-free graphs whenever F is not special. On the other hand, for every special graph F with at most four vertices, MWSS can be solved in polynomial time in the class of F-free graphs [5]. However, for most special graphs F, the complexity of MSS in the class of F-free graphs is still unknown. The path on five vertices P_5 is the smallest such graph. There are many results on the existence of polynomial-time algorithms for MSS in subclasses of P_5-free graphs; see for example [5, 12].

In any graph G, a k-coloring is a mapping $\gamma : V(G) \to \{1, \dots, k\}$ such that any two adjacent vertices u, v satisfy $\gamma(u) \neq \gamma(v)$. A graph G is called *k-colorable* if it admits a k-coloring. A graph is called *bipartite* if it is 2-colorable.

It is well-known that MWSS can be solved in polynomial time in the class of bipartite graphs, for in that case the problem is equivalent to the maximum matching problem [17]. Moreover, MWSS can be solved in linear time in the class of 2-colorable P_5-free graphs; see Section 2 below. Generalizing this idea, it was proved in [13] that, for fixed k, MWSS can be solved in polynomial time in the class of k-colorable graphs. More precisely, it was proved in [13] that if there exists an algorithm that solves MWSS in time $O(n^c)$ in $(k-1)$-colorable P_5-free graphs, then there exists an algorithm that solves MWSS in time $O(n^{c+1})$ in k-colorable P_5-free graphs (assuming that a k-coloring of the input graph is given; but note that, for fixed k, one can find a k-coloring in polynomial time in the class of P_5-free graphs, as proved in [9]). This shows that any improvement of the complexity of such an algorithm in k-colorable P_5-free graphs entails the same improvement in ℓ-colorable P_5-free graphs for all $\ell > k$.

Here we focus on 3-colorable P_5-free graphs. The result from [13] shows that MWSS can be solved in time $O(n^3)$ in that class, assuming that a 3-coloring is given. We will show that it can be solved in linear time in that class. Our result is based on a description of the structure of 3-colorable P_5-free graphs. It also implies that we can decide in linear time whether a given graph is in that class.

For any integer $h \geq 1$, let Γ_{3h+1} be the graph with $3h+1$ vertices $g_0, \dots, g_{3h}$ where the neighbors of g_i are $g_{i+h}, \dots, g_{i+2h+1}$ (with subscripts modulo $3h+1$). Now let $\mathcal{F} = \{F_1, \dots, F_{12}\}$ be the family of twelve graphs described as follows. Graph F_1 is Γ_4, which is the complete graph on four vertices. Graph F_2 is the "5-wheel", the graph with six vertices where the first five induce a cycle of length 5 and are all adjacent to the sixth vertex. Graph F_3 has vertices $u_1, \dots, u_5, v_2, v_5$, where $u_1, \dots, u_5$ induce a cycle of length 5 in this order, v_2 is adjacent to u_1, u_2, u_3, and v_5 is adjacent to u_4, u_5, u_1. Graph F_4 is F_3 plus the edge v_2v_5. Graph F_5 is F_3 plus the edge v_2u_4. Graph F_6 has vertices $v_1, \ldots, v_7$ and edges v_1v_2, v_2v_3, v_3v_4, v_4v_5, v_5v_6, v_6v_1, v_1v_3, v_3v_5, v_5v_1, v_7v_2, v_7v_4, v_7v_6. Graph F_7 is F_5 plus the edge v_5u_3. Graph F_8 is F_4 plus the edge u_2u_5. Graph F_9 is Γ_7, which is the complement of a cycle of length 7. Graph F_{10} has ten vertices $v_1, \ldots, v_6, x, x', y, y'$, where each v_i is adjacent to $v_{i+2}, v_{i+3}, v_{i+4}$ with subscripts modulo 6 (so $v_1, \ldots, v_6$ induce the complement of a cycle of length 6) and with edges xx', xv_1, $x'v_1$, xv_2, $x'v_2$, yy', yv_4, $y'v_4$, yv_5, $y'v_5$. Graph F_{11} is Γ_{10}. Graph F_{12} is Γ_{13}.

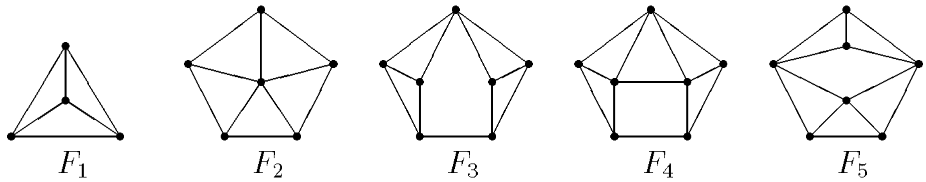

Figure 1: Graphs F_1–F_5.

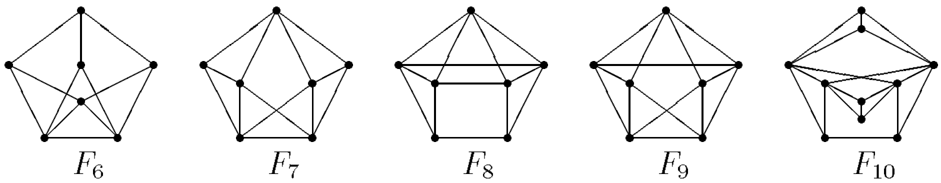

Figure 2: Graphs F_6–F_{10}.

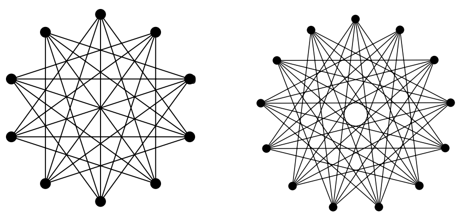

Figure 3: Graphs F_{11} and F_{12}

Let $\mathcal{C}$ be the class of 3-colorable P_5-free graphs. Our main results are the following three theorems.

Theorem 1.1. *A graph is in $\mathcal{C}$ if and only if it does not contain P_5 or any of $F_1, \ldots, F_{12}$ as an induced subgraph.*

Theorem 1.2. *One can decide in linear time whether a graph is in $\mathcal{C}$.*

Theorem 1.3. *One can solve MWSS in linear time in $\mathcal{C}$.*

Theorem 1.1 was proved by Bruce, Hoàng and Sawada [2] under a slightly different but equivalent form, as follows. Let $\mathcal{F}' = \{F_1, F_2, F_3, F_6, F_{10}, F_{12}\}$. (In [2], these graphs are called K_4, W_5, S_1, S_2, T and B, respectively.) Let us say that G *hosts* F when F appears as a *not necessarily induced* subgraph of G. With a harmless abuse, we may also say that a set $X \subseteq V(G)$ hosts F when $G[X]$ hosts F.

Theorem 1.4 ([2]). *A P_5-free graph is in $\mathcal{C}$ if and only if it does not host any member of $\mathcal{F}'$.*

The equivalence of Theorems 1.1 and 1.4 is clarified by the following lemma.

Lemma 1.5. *If a graph G hosts one of F_9, F_{11}, F_{12}, then it contains one of F_1, F_9, F_{11}, F_{12}. If a graph G hosts a member of $\mathcal{F}'$, then it contains a P_5 or a member of $\mathcal{F}$.*

Proof. First we prove the first sentence of the lemma. By the hypothesis $V(G)$ contains, for some $h \in \{2, 3, 4\}$, a set $X = \{x_0, \ldots, x_{3h}\}$ such that each x_i is adjacent to $x_{i+h}, \ldots, x_{i+2h+1}$ (with subscripts modulo $3h+1$), and there may be other edges in $G[X]$, which we call optional edges. If there is no optional edge, then $G[X]$ induces a Γ_h. Now suppose, without loss of generality, that there is an optional edge x_0x_j ($j < h$). If $j = 1$, then $\{x_0, x_1, x_{h+1}, x_{2h+1}\}$ induces a K_4 ($= F_1$). If $j = 2$, then $h \in \{3, 4\}$ and the set $\{x_0, x_1, x_2, x_{h+1}, x_{h+2}, x_{2h+1}, x_{2h+2}\}$ hosts a Γ_7 ($= F_9$), and so, by the preceding argument, G contains an F_9 or F_1. If $j = 3$, then $h = 4$ and the set $X \setminus \{x_4, x_8, x_{12}\}$ hosts a Γ_{10} ($= F_{11}$), and so, by the preceding arguments, G contains an F_{11}, F_9 or F_1.

Now we prove the second sentence of the lemma. Suppose that G hosts a member F of $\mathcal{F}'$. Then the following facts are easy to establish. If F is F_1 ($= K_4$), then G contains F_1. If F is F_2, then G contains either F_2 or F_1. If F is F_3, then G contains one of F_1, F_2, F_3, F_4, F_5, F_7, F_8, F_9. If F is F_6, then G contains one of F_1, F_2, F_6, F_8. If F is F_{10}, then G contains either a P_5 or some F_i with $1 \leq i \leq 10$. Finally, if F is F_{12}, then, by the first sentence of the lemma, G contains F_1, F_9, F_{11} or F_{12}. □

In the remainder of this paper, we give a common proof for Theorems 1.3–1.1. The proof is an algorithm that takes any arbitrary graph G_0 as input and, in linear time, returns one of the following outputs:
- The algorithm returns an induced subgraph of G_0 that is either a P_5 (so G_0 is not P_5-free) or a member of $\mathcal{F}$ (so G_0 is not 3-colorable);
- The algorithm returns a certain structural layout of G_0, from which we can, still in linear time, test whether G_0 is 3-colorable, compute a maximum weight stable set of G_0, and decide whether G is P_5-free.

The algorithm works roughly as follows. (Technical terms will be defined below.) First the algorithm eliminates homogeneous sets (modules) from the graph. Then it tests whether the complement of the graph is chordal. If it is chordal, our tasked is finished. If the complement of the graph is not chordal, the algorithm finds a set of vertices that induce a hole in the complement, and goes on to build the structure of the whole graph around that set. The rest of the paper is devoted to the precise presentation of the algorithm and the proof of its correctness.

We close this section with a few standard definitions. In a graph G, for any vertex v of G let $N(v)$ denote the set of all neighbors of v, and let $d(v)$ denote the *degree* of v, that is, the cardinality of $N(v)$. We say that a vertex v is *complete* to a set $X \subseteq V(G)$ if v is adjacent to every vertex of X. We say that v is *anticomplete* to X if v has no neighbor in X. Also we say that a set X is complete to a set Y (or that the two sets X, Y are complete to each other) if every vertex of X is complete to Y, and that X is anticomplete to Y if there is no edge between X and Y. We denote by K_n the complete graph on n vertices, by P_n the path on n vertices, and by C_n the cycle on n vertices. We let $2K_2$ denote the graph with four vertices $\{a, b, c, d\}$ and two edges ab and cd.

2 Bipartite P_5-free graphs

A graph G is called *bipartite* if its vertex-set can be partitioned into two stable sets A, B. In that case we write $G = (A, B; E)$. The structure of bipartite P_5-free graphs is very useful to us. It is completely elucidated in the following classical theorems. See in particular [15, Section 2.4] and [7] for more details.

Theorem 2.1. *A connected bipartite graph G is P_5-free if and only if it is $2K_2$-free.*

Proof. Clearly, if G contains no $2K_2$ then it contains no P_5. Conversely, suppose that G contains a $2K_2$ with vertices a, b, c, d and edges ab, cd. Since G is connected, there is a shortest path P from $\{a, b\}$ to $\{c, d\}$ in G. Then it is a routine matter to check that $V(P) \cup \{a, b, c, d\}$ contains five vertices that induce a P_5 in G. □

The preceding theorem implies that, in a bipartite P_5-free graph $G = (A, B, E)$, any two vertices u, and v in A satisfy either $N(u) \subseteq N(v)$ or $N(v) \subseteq N(u)$, i.e., the vertices of A can be totally ordered by neighborhood inclusion; and the same holds for B. This implies the following theorem.

Theorem 2.2. *Let $G = (A, B; E)$ be a connected P_5-free bipartite graph with $E \neq \emptyset$. Then, for some integer $h \geq 1$, there exists a partition $A_1, \ldots, A_h$ of A into h non-empty sets and a partition $B_1, \ldots, B_h$ of B into h non-empty sets such that a vertex in A_i and a vertex in B_j are adjacent if and only if $i + j \leq h + 1$. Consequently, G has exactly $h + 1$ maximal stable sets, which are A, B and $A_{i+1} \cup \cdots \cup A_h \cup B_{h+1-i} \cup \cdots \cup B_h$ for each $i = 1, \ldots, h - 1$.*

From an algorithmic point of view, testing that a given connected bipartite graph $G = (A, B; E)$ is P_5-free can be done as follows.

> While A has a vertex a of degree $|B|$, then remove a from A (and update the degree of each vertex of B). While B has a vertex b of degree 0, then remove b from B.
> If this procedure does not exhaust the sets A and B, then declare that G is not P_5-free. (This is correct because of the following argument. Suppose that A, B are non-empty at the end of the procedure. Pick a vertex a in A of maximum degree; then $d(a) < |B|$, so there is a vertex b in $B \setminus N(a)$. The procedure implies that b has a neighbor u in A, and the choice of a implies that there is a vertex v in B adjacent to a and not to u. Then $\{a, u, b, v\}$ induces a $2K_2$, so, by Theorem 2.1, G contains a P_5.)
> Else declare that G is P_5-free. This is correct by the preceding theorem.

Clearly, this procedure can be implemented in linear time. Moreover, computing a maximum weight stable set can be done in linear time simply by examining the $h + 1$ stable sets mentioned at the end of Theorem 2.2.

In addition, we note the following easy corollary of the above theorems. Its proof is obvious and we omit it.

Theorem 2.3. *A bipartite graph is $2K_2$-free if and only if it has at most one component of size at least two and that component (if any) is P_5-free.*

Consequently, testing if a bipartite graph is $2K_2$-free can also be done in linear time.

3 Data structure

Throughout the algorithm, we will have to handle the following kinds of situation in a graph G. Assume that G has n vertices and m edges.

(a) Given some (pairwise disjoint) subsets $X_1, \ldots, X_\ell$ of $V(G)$, which may grow during the execution of the algorithm, we want to know for each vertex x of $V(G) \setminus (X_1 \cup \cdots X_\ell)$ whether x is anticomplete to X_i or not, for every $i \in \{1, \ldots, \ell\}$. This information can be recorded by a counter $c_i(x)$ equal to 0 or 1 respectively. The vector $(c_1(x), \ldots, c_\ell(x))$ is called the *type* of x with respect to the collection $X_1, \ldots, X_\ell$. Since ℓ will always be a constant, there is a constant number of different types. Each vertex x of $V(G) \setminus (X_1 \cup \cdots \cup X_\ell)$ is placed in a set Q_t associated with the type t of x. Whenever a vertex u of $V(G) \setminus (X_1 \cup \cdots \cup X_\ell)$ is added to a set X_i, we scan the adjacency list of u and, for each neighbor x of u with $c_i(x) = 0$, we set $c_i(x) = 1$ and move x to the set associated with its new type. This takes time $O(d(u))$. We can also decide whether there exists a vertex of any given type t in constant time by checking the corresponding set Q_t, and we can obtain all such vertices in time $O(|Q_t|)$.

(b) Given a set $X \subset V(G)$ (which will not change), we want to determine for every vertex u of $V(G) \setminus X$ whether u is complete to X, anticomplete to X, or neither. This can be done as follows. For each vertex x in X, scan the adjacency list of x, and for each neighbor u of x add 1 to a counter $c_X(u)$ (initialized at value 0). When this is done for all vertices of X, compare the final value of $c_X(u)$ with $|X|$. This takes time $O(m+n)$ and will be executed for only a constant number of sets X, so the total time is linear.

These remarks ensure that each main step in our algorithm has complexity $O(n+m)$. They will not be repeated explicitly throughout the algorithm.

4 Initialization

Let G_0 be any graph given as input of our algorithm. Let G_0 have n vertices and m edges. We may assume that G_0 is connected, else it suffices to process each component of G_0 separately.

A *homogeneous set* is a set $S \subseteq V(G_0)$ such that every vertex of $V(G_0) \setminus S$ is either complete or anticomplete to S. A *module* is a homogeneous set M

such that every homogeneous set S satisfies either $S \subseteq M$, or $M \subseteq S$, or $S \cap M = \emptyset$. Note that $V(G_0)$ and each $\{v\}$ ($v \in V(G_0)$) is a module of G_0. Say that a module M is *maximal* if $M \neq V(G_0)$ and there is no module M' with $M \subset M' \subset V(G_0)$ where the two inclusions are strict. The study of homogeneous sets and modules in a graph is a rich one, starting with Gallai's pioneering work [3, 14], leading to the so-called modular decomposition of a graph. Without going into the complex details, let us only mention the facts that we will use here.

- Every graph G_0 with n vertices and m edges has $O(n)$ modules, and they can be determined by an algorithm [20] of complexity $O(n+m)$.
- If G_0 has at least two vertices, then it has at least two maximal modules and the maximal modules of G_0 form a partition $\mathcal{M}$ of $V(G_0)$.

Thus we can obtain the partition $\mathcal{M}$ in time $O(n+m)$.

Lemma 4.1. *If G_0 has at least two vertices and any element Y of $\mathcal{M}$ does not induce a bipartite graph, then G_0 is not 3-colorable.*

Proof. Since G_0 is connected, there is a vertex u in $V(G_0)\backslash Y$ that is complete to Y. If Y does not induce a bipartite subgraph, then it contains the vertex-set Z of an odd cycle. Then $Z \cup \{u\}$ induces a subgraph of G_0 that is not 3-colorable. Actually, either Z has length at least 7, so it contains a P_5, or Z has length 3 or 5, and then $Z \cup \{u\}$ induces an F_1 or F_2. □

Our algorithm tests if all elements of $\mathcal{M}$ induce bipartite subgraphs. Since they are disjoint, this takes time $O(n+m)$. If any of them does not induce a bipartite subgraph, the algorithm returns a P_5 or a member of $\mathcal{F}$ as shown in the preceding proof and stops. Now let us assume that every Y in $\mathcal{M}$ induces a bipartite subgraph. Let G be the graph obtained from G_0 by "contracting" each element of $\mathcal{M}$, that is, for each $Y \in \mathcal{M}$, choose one vertex y of Y and remove all vertices of $Y \setminus y$. We call G the *reduced graph* of G_0. If an element Y of $\mathcal{M}$ is not a stable set, then we choose a vertex y that has a neighbor $\overline{y}$ in Y, and in G we say that y is a *double* vertex. Note that y and $\overline{y}$ have the same neighbors in $V(G_0) \setminus Y$. Whenever we handle such a vertex y below, we will remember that it represents a bipartite subgraph of G_0 with non-empty edge-set, and in particular that it needs two colors. So G_0 is 3-colorable if and only if G admits a 3-coloring such that (with a slight abuse of the definition) every double vertex has two colors, which are different from the colors of its neighbors. A vertex that is not double will be called *simple*.

Lemma 4.2. *Graphs G_0 and G satisfy the following properties.*
(i) Either G is a clique or G has no homogeneous set.
(ii) G_0 is P_5-free if and only if G and all subgraphs $G_0[Y]$ ($Y \in \mathcal{M}$) are P_5-free.
(iii) One can deduce in linear time a maximum weight stable set of G_0 from any maximum weight stable set of G.

Proof. Fact (i) follows from modular decomposition theory and we do not repeat its proof. See for example [3, 14, 20].

(ii) If G_0 is P_5-free, then so is G, since G is an induced subgraph of G_0. Conversely, suppose that G_0 contains a P_5 with vertex set X. Since P_5 has no homogeneous set, it must be that either X is included in a member Y of $\mathcal{M}$, so $G_0[Y]$ contains a P_5, or X has at most one vertex from each member of $\mathcal{M}$, and so X induces a P_5 in G.

(iii) Let G have vertices $y_1, \ldots, y_p$, and let Y_i be the member of $\mathcal{M}$ that is represented by y_i for each $i \in \{1, \ldots, p\}$. For each $i \in \{1, \ldots, p\}$, let S_i be a maximum weight stable set in the subgraph $G_0[Y_i]$ with weight function the restriction of f to Y_i. If S is any stable set in G, then the set $U(S) = \bigcup_{y_i \in S} S_i$ is a stable set in G_0. Conversely every stable set in G_0 is included in a set of the form $\bigcup_{y_i \in S} T_i$ where T_i is a stable set in $G_0[Y_i]$. It follows that if S is a maximum weight stable set in G, then $U(S)$ is a maximum weight stable set in G_0. □

5 Chordality of $\overline{G}$

In this step, we test whether the complement $\overline{G}$ of G is chordal; and if it is chordal, we compute $\omega(G)$ and $\alpha_f(G)$. Testing if a graph is chordal can be done in linear time as shown in [18]; in addition, when the graph is not chordal, an induced cycle can be reconstructed in linear time as explained in [19]. However, we do not want to look at the complementary graph of our graph G, since its construction may take time $O(n^2)$. One can test directly whether G is the complement of a chordal graph in time $O(n+m)$ as proved in [8]. The reconstruction step is not given explicitly in [8], but it can be implemented in linear time similarly to [19]. This procedure has the following three possible outcomes:

(1) $\overline{G}$ is not chordal. Then the algorithm in [8] returns a subset of vertices Z that induces a cycle in $\overline{G}$. If $|Z| = 4$, then $G[Z]$ is a $2K_2$ and our algorithm goes to Section 6. If $|Z| = 5$ or 6, then $G[Z]$ is a C_5 or $\overline{C_6}$ and our algorithm goes to Section 7 or 8 respectively. If $|Z| = 7$, then our algorithm returns $G[Z]$,

which is a $\overline{C_7}$ (a member of $\mathcal{F}$), and stops. If $|Z| \geq 8$, then $G[Z]$ contains a K_4; our algorithm returns this K_4 (a member of $\mathcal{F}$) and stops.

(2) $\overline{G}$ is chordal and $\omega(G) \geq 4$. Then our algorithm returns the message "G contains a K_4" and stops.

(3) $\overline{G}$ is chordal and $\omega(G) \leq 3$. Then our algorithm returns the message "G is P_5-free." This is a correct answer because the complement of a chordal graph is clearly P_5-free. There remains to know whether G_0 is 3-colorable or not. This can be done as follows. Define a new weight function by assigning weight 2 to the double vertices of G and weight 1 to the other vertices. Then find the maximum clique size in G with that weight function. This can be done by the algorithm from [8]. Clearly, G_0 is 3-colorable if and only if the maximum weight clique in G has weight 3. The algorithm from [8] computes a maximum weight clique and a minimum weight coloring of the weighted graph, that is, a collection $S_1, \ldots, S_k$ of stable sets with corresponding weights $s_1, \ldots, s_k$ such that every vertex x satisfies $w(x) \leq \sum\{s_i \mid S_i \ni x\}$. It is shown in [8] that the complexity of this algorithm is bounded by $\bigcup_{i=1}^{k} |S_i|$. Since our vertices have weight 1 or 2, a simple analysis of the algorithm from [8] shows that each S_i has weight 1 or 2 and each vertex of G appears in at most two of $S_1, \ldots, S_k$. It follows that the sum $\bigcup_{i=1}^{k} |S_i|$ is bounded by $2n$, so the complexity is linear.

6 G contains a $2K_2$

Let H_1 be the graph with six vertices u, v, w, x, y, z and eight edges $uv, uw, vw, xy, xz, yz, vy, wz$. Let H_2 be the graph H_1 plus the edge vz.

The following lemma appeared in [11, 10] under a different form.

Lemma 6.1. *If G contains a $2K_2$, then G contains either a P_5 or one of H_1 or H_2.*

Proof. This proof is an algorithm that finds one of the three subgraphs P_5, H_1 or H_2. Let a_1, a_2, b_1, b_2 be four vertices that induce a $2K_2$ in G, with edges a_1a_2 and b_1b_2. Let $A = \{a_1, a_2\}$ and $B = \{b_1, b_2\}$. Apply the following procedure.

> While there is a vertex x in $V(G) \setminus (A \cup B)$ that has a neighbor in A and no neighbor in B, then add x to A.
> While there is a vertex y in $V(G) \setminus (A \cup B)$ that has a neighbor in B and no neighbor in A, then add y to B.
> Repeat these two operations as long as possible.

Now consider the sets A and B at the end of this procedure. Since G is a reduced graph, A is not a homogeneous set, so there exists a vertex x in $V(G) \setminus A$ that is not complete or anticomplete to A, i.e., there are two adjacent vertices a and a' in A such that x is adjacent to a and not to a'. Clearly, x is not in B, because the vertices in B are anticomplete to A. Likewise, there exists a vertex y in $V(G) \setminus B$ and two adjacent vertices b, b' in B such that y is adjacent to b and not to b', and y is not in A. If x is adjacent to exactly one of b, b', then $\{a, a', x, b, b'\}$ induces a P_5. If x is not adjacent to any of b, b', then, by the definition of the procedure, x has a neighbor c in B (else x would have been added to A), and there is a shortest path P in B between c and $\{b, b'\}$. Then $\{a, a', x\} \cup V(P)$ contains a P_5. Now assume that x is adjacent to both b, b'. Likewise, assume that y is adjacent to both a, a'. Then $\{a, a', b, b', x, y\}$ induces an H_1 (if $xy \notin E(G)$) or an H_2 (else). □

From an algorthmic point of view, it is easy to construct the sets A and B starting from $\{a_1, a_2\}$ and $\{b_1, b_2\}$ and to obtain the vertices x and y, by applying remarks (a) and (b) given in Section 3. So the total complexity of this step is $O(m + n)$.

If the algorithm returns an H_1, we go to Section 6.1; if it returns an H_2, we go to Section 6.2.

6.1 G contains an H_1

Let $a_1, \dots, a_6$ be six vertices of G that induce an H_1, with edges a_1a_2, a_1a_3, a_2a_3, a_4a_5, a_4a_6, a_5a_6, a_2a_4, a_3a_5. Let $A' = \{a_1, a_2, a_3\}$, $A'' = \{a_4, a_5, a_6\}$, and $A = A' \cup A''$.

Lemma 6.2. *Either G has only six vertices, of G contains a P_5 or a member of $\mathcal{F}$.*

Proof. Suppose that G has at least seven vertices. Since G is connected, there is a vertex u in $V(G) \setminus A$ that has a neighbor in A. If u is adjacent to all vertices of A' or all vertices of A'', then G contains a K_4. Now assume, up to symmetry, that u has one or two neighbors in A'.

Suppose that $N(u) \cap A' = \{a_1\}$. If u is not adjacent to a_4 or a_5, then u-a_1-a_2-a_4-a_5 is an induced P_5. So assume, up to symmetry, that u is adjacent to a_4. If u is not adjacent to a_6, then a_6-a_4-u-a_1-a_3 is a P_5. If u is adjacent to a_6, then, as mentioned above, u is not adjacent to a_5, and then a_5-a_6-u-a_1-a_2 is a P_5.

Suppose that $N(u) \cap A' = \{a_2\}$. If u has no neighbor in $\{a_5, a_6\}$, then u-a_2-a_3-a_5-a_6 is an induced P_5. If u is adjacent to exactly one of a_5 or a_6, then $\{u, a_1, a_2, a_5, a_6\}$ induces a P_5. If u is adjacent to both a_5 and a_6, then, as mentioned above, u is not adjacent to a_4. Thus $N(u) \cap A = \{a_2, a_5, a_6\}$. Let $X = \{x \in V(G) \mid N(x) \cap A = \{a_2, a_5, a_6\}\}$. We know that $|X| \geq 2$ since a_4 and u are in X. Since G is a reduced graph, X is not a homogeneous set, so there are vertices v in $V(G) \setminus X$ and x, y in X such that v is adjacent to x and not to y. Clearly, $v \notin A$. If v is adjacent to any $a \in \{a_1, a_3\}$, then a-v-x-a_6-y is a P_5. So let $va_1, va_3 \notin E$. If v is not adjacent to a_5, then v-x-a_5-a_3-a_1 is a P_5. So let $va_5 \in E$. If v is not adjacent to a_2, then v-a_5-y-a_2-a_1 is a P_5. So let $va_2 \in E$. If v is not adjacent to a_6, then a_6-a_5-v-a_2-a_1 is a P_5. So let $va_6 \in E$. Then $\{v, x, a_5, a_6\}$ induces a K_4.

The proof is similar if $N(u) \cap A' = \{a_3\}$.

We may now assume that u has two neighbors in A'. By symmetry, we may assume that u has either zero or two neighbors in A''.

Suppose that $N(u) \cap A' = \{a_1, a_2\}$. If u is adjacent to both a_4 and a_6, then $A \cap \{u\}$ induces an F_4. If u is adjacent to exactly one of a_4 or a_6, then $\{u, a_1, a_3, a_4, a_6\}$ induces a P_5. So assume that u is not adjacent to any of a_4 and a_6. Since u has zero or two neighbors of A'', it is not adjacent to a_5. Then u-a_2-a_3-a_5-a_6 is a P_5. The proof is similar if $N(u) \cap A' = \{a_1, a_3\}$.

Now we may assume that $N(u) \cap A' = \{a_2, a_3\}$ and, by symmetry, $N(u) \cap A''$ is either equal to $\{a_4, a_5\}$ or empty. In the first case a_1-a_2-u-a_5-a_6 is an induced P_5. In the second case, we have $N(u) \cap A = \{a_2, a_3\}$. Let then $X = \{x \in V(G) \mid N(x) \cap A = \{a_2, a_3\}\}$. We know that $|X| \geq 2$ since a_1 and u are in X. Since G is a reduced graph, X is not a homogeneous set, so there are vertices v in $V(G) \setminus X$ and x, y in X such that v is adjacent to x and not to y. Clearly, $v \notin A$. If v is adjacent to any $a \in \{a_5, a_6\}$, then a-v-x-a_2-y is a P_5. So let $va_5, va_6 \notin E$ and, by symmetry, $va_4 \notin E$. If v is not adjacent to a_2, then v-x-a_2-a_4-a_6 is a P_5. So let $va_2 \in E$ and, by symmetry, $va_3 \in E$. Then $\{v, x, a_2, a_3\}$ induces a K_4. This completes the proof of the lemma. □

If the algorithm has not found a P_5 or a member of $\mathcal{F}$, then G has only six vertices. Clearly, G is P_5-free. Moreover, solving the maximum weighted stability problem is trivial. In addition, we note that every vertex of G lies in a triangle. So if any such vertex is a double vertex, then G_0 contains a K_4. If G has no double vertex, then each vertex a_i of G represents a stable set Y_i (a member of $\mathcal{M}$) in G_0, and, G_0 is 3-colorable; it suffices to assign color 1 to $Y_1 \cup Y_6$, color 2 to $Y_2 \cup Y_5$, and color 3 to $Y_3 \cup Y_4$.

As a final note for this section, we remark that, when G contains an H_1 and does not contain a P_5 or a member of $\mathcal{F}$, then G does not contain an induced C_5 or $\overline{C}_6$.

6.2 G contains an H_2

Let $a_1, \ldots, a_6$ be six vertices of G that induce an H_2, with edges a_1a_2, a_1a_3, a_2a_3, a_4a_5, a_4a_6, a_5a_6, a_2a_4, a_3a_4, a_3a_5.

Let us say that a graph H has an *H_2-structure* if its vertex-set $V(H)$ can be partitioned into six non-empty sets $A_1, \ldots, A_6$ such that:

- Each of $A_1, \ldots, A_6$ is a stable set,
- A_3 is complete to $A_1 \cup A_2 \cup A_4 \cup A_5$ and anticomplete to A_6,
- A_4 is complete to $A_2 \cup A_3 \cup A_5 \cup A_6$ and anticomplete to A_1,
- $A_1 \cup A_2$ is anticomplete to $A_5 \cup A_6$,
- Every vertex of A_1 has a neighbor in A_2 and every vertex of A_2 has a neighbor in A_1,
- Every vertex of A_6 has a neighbor in A_5 and every vertex of A_5 has a neighbor in A_6.

Note that the six vertices $a_1, \ldots, a_6$ induce a graph with an H_2-structure ($\{a_1\}$, $\ldots$, $\{a_6\}$). Our aim is to "grow" it to the largest possible structure.

Let H be an induced subgraph of G with an H_2-structure ($A_1, \ldots, A_6$), and x be a vertex of $V(G) \setminus V(H)$. We say that x can be *added* to this H_2-structure if the subgraph induced by $V(H) \cup \{x\}$ has an H_2-structure $(A'_1, \ldots, A'_6)$ where $A'_j = A_j \cup \{x\}$ for some $j \in \{1, \ldots, 6\}$ and $A'_i = A_i$ for all $i \neq j$.

Lemma 6.3. *Let H be an induced subgraph of G with an H_2-structure $(A_1, \ldots, A_6)$. Let x be a vertex of $V(G) \setminus V(H)$ that has a neighbor in $A_1 \cup A_2 \cup A_5 \cup A_6$. Then either x lies in a subgraph of G that induces a P_5 or a member of $\mathcal{F}$, or x can be added to the H_2-structure $(A_1, \ldots, A_6)$.*

Proof. The proof is an algorithm that determines which outcome holds. Up to symmetry, we may assume that x has a neighbor in $A_1 \cup A_2$.

Case 1: x has a neighbor u_1 in A_1.
Let u_2 be any neighbor of u_1 in A_2. Such a vertex exists by the definition of an H_2-structure.

First suppose that x has a neigbhor u_3 in A_3. If x is adjacent to u_2, then $\{x, u_1, u_2, u_3\}$ induces a K_4. Now let $xu_2 \notin E$. Pick any u_4 in A_4 and u_5 in

A_5. If x is adjacent to both u_4, u_5, then $\{x, u_3, u_4, u_5\}$ induces a K_4. If x is not adjacent to any of u_4, u_5, then x-u_1-u_2-u_4-u_5 is an induced P_5. If x is adjacent to u_5 and not to u_4, then $\{x, u_1, u_2, u_3, u_4, u_5\}$ induces an F_2. Therefore let us assume that x is complete to A_4 and anticomplete to A_5. If x has a neighbor u_6 in A_6, then u_2-u_1-x-u_6-u_5 is a P_5, for any u_5 in $A_5 \cap N(u_6)$. Thus let us assume that x is anticomplete to A_6. If x has a non-neighbor v_3 in A_3, then x-u_1-v_3-u_5-u_6 is a P_5, for any u_5 in A_5 and u_6 in $A_6 \cap N(u_5)$. If x has any neighbor v_2 in A_2, then then $\{x, v_2, u_3, u_4\}$ induces a K_4 for any u_4 in A_4. Now x is complete to $A_3 \cup A_4$, anticomplete to $A_2 \cup A_5 \cup A_6$, and x has a neighbor in A_1, so x can be added to the H_2-structure (x can be placed in A_2).

Now assume that x has no neighbor in A_3. Pick any u_3 in A_3, u_4 in A_4, u_5 in A_5, and u_6 in $A_6 \cap N(u_5)$. Suppose that x is not adjacent to u_2. If x is not adjacent to any of u_5, u_6, then x-u_1-u_3-u_5-u_6 is a P_5. If x is adjacent to exactly one of u_5, u_6, then $\{x, u_1, u_2, u_5, u_6\}$ induces a P_5. So let x be adjacent to both u_5, u_6. If x is adjacent to u_4, then $\{x, u_4, u_5, u_6\}$ induces a K_4. If x is not adjacent to u_4, then $\{x, u_1, \dots, u_6\}$ induces an F_4. Therefore assume that x is adjacent to u_2. Suppose that x has a neighbor u_6 in A_6, and pick any u_5 in $A_5 \cap N(u_6)$. If x is not adjacent to u_4, then $\{x, u_1, \dots, u_6\}$ induces an F_4 or F_7 (depending on the existence of xu_5). If x is adjacent to u_4 and not to u_5, then $\{x, u_2, \dots, u_6\}$ induces an F_2. If x is adjacent to u_4 and u_5, then $\{x, u_4, u_5, u_6\}$ induces a K_4. Now assume that x has no neighbor in A_6. If x has a non-neighbor w in $A_4 \cup A_5$, then x-u_1-u_2-w-u_6 is a P_5, for any u_6 in $A_6 \cap N(w)$. So assume that x is complete to $A_4 \cup A_5$. If x has a neighbor v_3 in A_3, then $\{x, v_3, u_4, u_5\}$ induces a K_4 for any u_4 in A_4 and u_5 in A_5. So assume x has no neighbor in A_3. Suppose x has a non-neighbor v_2 in A_2. If v_2 is adjacent to u_1, then we can argue as at the beginning of this paragraph (with u_2). So assume that v_2 is not adjacent to u_1. By the definition of an H_2-partition, v_2 has a neighbor v_1 in A_1, and by the same argument we may assume that x is not adjacent to v_1. Then u_1-x-u_4-v_2-v_1 is a P_5. So assume that x is complete to A_2. Suppose that x has a non-neighbor w_1 in A_1. Let w_2 be any neighbor of w_1 in A_2. Then w_1-w_2-x-u_5-u_6 is a P_5, for any u_5 in A_5 and u_6 in $A_6 \cap N(u_5)$. Now x is complete to $A_1 \cup A_2 \cup A_4 \cup A_5$ and anticomplete to A_6, so so x can be added to the H_2-structure (x can be placed in A_3).

Case 2: x has a neighbor u_2 in A_2.
Since we are not in case 1, and by symmetry, we may assume that x has no neighbor in $A_1 \cup A_6$. If x has a neighbor u_5 in A_5, then u_1-u_2-x-u_5-u_6 is a P_5, for any u_1 in $A_1 \cap N(u_2)$, u_5 in A_5 and u_6 in $A_6 \cap N(u_5)$. So assume that x

has no neighbor in A_5. If x has a non-neighbor u_3 in A_3, then x-u_2-u_3-u_5-u_6 is a P_5. So assume that x is complete to A_3. If x has a neighbor u_4 in A_4, then $\{x, u_2, u_3, u_4\}$ induces a K_4 for any u_3 in A_3. So assume that x has no neighbor in A_4. Now x is complete to A_3, anticomplete to $A_1 \cup A_4 \cup A_5 \cup A_6$, and x has a neighbor in A_2, so x can be added to the H_2-structure (x can be placed in A_1). □

From an algorithmic point of view, it is easy to test the properties of the preceding lemma. Initially we set $A_i = \{a_i\}$ for each $i \in \{1, \ldots, 6\}$. Then we apply remarks (a) and (b) from Section 3. So the total time is $O(m + n)$.

Now we may assume that we have an H_2-structure $(A_1, \ldots, A_6)$ such that no vertex of $V(G) \setminus (A_1 \cup \cdots \cup A_6)$ has a neighbor in $A_1 \cup A_2 \cup A_5 \cup A_6$. Let $A = A_1 \cup \cdots \cup A_6$ and $R = V(G) \setminus A$.

Lemma 6.4. *If G contains no P_5, then every vertex of R has a neighbor in $A_3 \cup A_4$. Moreover, R is a stable set.*

Proof. First suppose that some vertex x in R has no neighbor in A. Since G is connected, there is a shortest path p_0-$\cdots$-p_k between x and A, that is, $x = p_0$, $p_0, \ldots, p_{k-1} \in V(G) \setminus A$ and $p_k \in A$, with $k \geq 2$. By the definition of R and up to symmetry, we may assume that $p_k \in A_3$. Pick any u_5 in A_5 and u_6 in $A_6 \cap N(u_5)$. If $k \geq 4$, then x-p_1-p_2-p_3-p_4 is a P_5. If $k = 3$, then x-p_1-p_2-p_3-u_5 is a P_5. If $k = 2$, then x-p_1-p_2-u_5-u_6 is a P_5.

Now suppose that R is not a stable set, and let Y be the vertex-set of a component of $G[R]$ of size at least two. Since G is a reduced graph, Y is not a homogeneous set, so there are adjacent vertices y, z in Y and a vertex x in $V(G) \setminus Y$ that is adjacent to y and not to z. By the definition of R and Y, and up to symmetry, let $x \in A_3$. Pick any u_5 in A_5 and u_6 in $A_6 \cap N(u_5)$. Then z-y-x-u_5-u_6 is a P_5. □

Let R' be the set of vertices of R that have a neighbor in A_3 and a neighbor in A_4. For $j \in \{3, 4\}$, let R_j be the set of vertices of $R \setminus R'$ that have a neighbor in A_j. So $R = R_3 \cup R_4 \cup R'$.

Lemma 6.5. *G_0 is 3-colorable if and only if $A \cup R'$ contains no double vertex.*

Proof. Suppose that $A \cup R'$ contains a double vertex x. By the definition of A_1, $\ldots$, A_6, R', it is easy to see that x has two adjacent neighbors u, v in G. Then $\{x, \overline{x}, u, v\}$ induces a K_4 in G_0, so G_0 is not 3-colorable.

Now suppose that $A \cup R'$ does not contain any double vertex. Assign color 1 to the vertices of $A_1 \cup A_4$, color 2 to the vertices of $A_2 \cup A_5$, color 3 to the vertices of $A_3 \cup A_6$; if u is a double vertex in R_3, assign colors 1 and 2 to u, and if u is a double vertex in R_4, assign colors 2 and 3 to u; finally, assign color 2 to the remaining (uncolored) vertices of R. It is easy to check that this is a 3-coloring of G and to derive from this a 3-coloring of G_0. □

Lemma 6.6. *Every stable set of G is a subset of one of the following three sets: $V_0 = A_1 \cup A_2 \cup A_5 \cup A_6 \cup R$, $V_3 = A_3 \cup A_6 \cup R$, $V_4 = A_4 \cup A_1 \cup R$. Moreover, each of these three sets induces a bipartite subgraph of G.*

Proof. Let S be any stable set in G. Suppose that S contains a vertex of A_3. Then the definition of an H_2-structure implies that S contains no vertex of $A_1 \cup A_2 \cup A_4 \cup A_5$. So $S \subseteq V_3$. Likewise, if S contains a vertex of A_4, then $S \subseteq V_4$. On the other hand, if S contains no vertex of $A_3 \cup A_4$, then $S \subseteq V_0$. This proves the first sentence of the lemma. Moreover, we observe that V_3 induces a bipartite subgraph of G because $A_3 \cup A_6$ and R are two stable sets; similarly V_4 induces a bipartite subgraph of G; and V_0 induces a bipartite subgraph of G because $A_1 \cup A_6 \cup R$ and $A_2 \cup A_5$ are two stable sets. □

It follows from the preceding lemma that, in order to solve MWSS for G, it suffices to solve it for the three bipartite subgraphs $G[V_0]$, $G[V_3]$, $G[V_4]$ and to return the best solution among the three. By the results of Section 2, this can done in linear time.

The algorithm has reached this point without finding a P_5 and has successfully solved the maximum weighted stable set problem in linear time, but that does not mean that the graph does not contain any P_5. We can clarify this question by the following lemma.

Lemma 6.7. *G is P_5-free if and only if each of the four subgraphs $G[A_1 \cup A_2]$, $G[A_5 \cup A_6]$, $G[A_3 \cup R]$, $G[A_4 \cup R]$ is $2K_2$-free.*

Proof. If either $G[A_1 \cup A_2]$ or $G[A_3 \cup R]$ contains a $2K_2$, then, by adding to the $2K_2$ any vertex from A_4, we obtain a P_5. The same holds (by adding a vertex from A_5) for $G[A_5 \cup A_6]$ and $G[A_4 \cup R]$.

Conversely, let x_1-x_2-x_3-x_4-x_5 be a P_5 in G. If none of $x_1, \ldots, x_5$ is in $A_3 \cup A_4$, then the P_5 lies in one of the subgraphs $G[A_1 \cup A_2]$ or $G[A_5 \cup A_6]$, so that subgraph is not $2K_2$-free. Now assume, up to symmetry, that one of $x_1, \ldots, x_5$ is in A_3. Suppose that $x_1 \in A_3$. Then its non-neighbors x_4, x_5 are in $A_3 \cup A_6 \cup R$, and, since they are adjacent, we have either $x_4 \in A_3$ and $x_5 \in R$ or $x_5 \in R$ and $x_4 \in A_3$. In either case, since x_2 is adjacent to x_1 and not to x_4 or x_5, it must be in R, and so $G[A_3 \cup R]$ contains a $2K_2$ with vertices x_1, x_2, x_4, x_5.

The same argument holds if $x_2 \in A_3$. Finally suppose that $x_3 \in A_3$. Then it is a routine matter to check that either $x_2, x_4 \in R$ and $x_1, x_5 \in A_3$ or $x_2, x_4 \in A_5$ and $x_1, x_5 \in A_6$, and so $G[A_3 \cup R]$ or $G[A_5 \cup A_6]$ contains a $2K_2$ with vertices x_1, x_2, x_4, x_5. □

It follows from the preceding lemma that, in order to verify that G is P_5-free, it suffices to verify that each of the above four bipartite subgraphs is $2K_2$-free. By the results of Section 2, this can done in linear time. Thus Lemmas 6.5–6.7 prove Theorems 1.3–1.1 is case G contains an H_2.

As a final note for this section, we remark that, when G contains an H_2 and does not contain a P_5 or a member of $\mathcal{F}$, then G does not contain an induced C_5 of $\overline{C}_6$. (This is easy to check and we omit the details.)

7 G contains a C_5

In this section, we assume that G contains a C_5, output by the algorithm in Section 5.

We note that if in this section the algorithm finds an induced $2K_2$, then we can go to Section 6 and start again from there. (In fact, we observed in the preceding sections that when G contains a $2K_2$ then either it is not in $\mathcal{C}$ or it contains no C_5. In other words, an alternate and valid action here is to stop with the answer "G is not in $\mathcal{C}$"; we need to apply the preceding step only to produce explicitly a forbidden subgraph.)

Let $a_1, \ldots, a_5$ be five vertices that induce a C_5 in G, with edges $a_i a_{i+1}$ (with subscripts modulo 5). Let us "grow" this C_5 to a larger structure, as follows. For each $i \in \{1, \ldots, 5\}$, let $R_i = \{a_i\}$, and let $R = R_1 \cup \cdots \cup R_5$. As explained in remark (a) in Section 3, every vertex $x \in V(G) \setminus R$ has a type $(c_1(x), \ldots, c_5(x))$, where $c_i(x)$ is equal to 0 if x is anticomplete to R_i and else is equal to 1. Apply the following procedure:

> While there is a vertex x in $V(G) \setminus R$ that is anticomplete to $R_{i-2} \cup R_{i+2}$ for some $i \in \{1, \ldots, 5\}$ and not anticomplete to R, test whether x is complete to $R_{i-1} \cup R_{i+1}$:
> If x is not complete to $R_{i-1} \cup R_{i+1}$, then returns the answer "G contains a P_5" and stop;
> else, add x to R_i, update the type of every neighbor of x, and continue.

Finding such vertices and testing them can be done in time $O(n+m)$ by remarks (a) and (b) in Section 3. We claim that when the procedure returns the answer "G contains a P_5", this is a correct answer. Indeed, consider the corresponding vertex x. There is a neighbor y of x in R, and y in $R_{i-1} \cup R_i \cup R_{i+1}$ since x is anticomplete to $R_{i-2} \cup R_{i+2}$. Also there is a non-neighbor z of x in $R_{i-1} \cup R_{i+1}$; up to symmetry, let $z \in R_{i-1}$. Pick any r_j in R_j for each $j \neq i-1$. If $y \in R_{i+1}$, then x-y-r_{i+2}-r_{i-2}-z is a P_5. If $y \in R_i$, then x-y-z-r_{i-2}-r_{i+2} is a P_5. If $y \in R_{i-1}$, then either x-y-r_{i-2}-r_{i+2}-r_{i+1} or x-r_{i+1}-r_{i+2}-r_{i-2}-z is a P_5. This proves our claim.

Now assume that the algorithm has not stopped with the answer that G contains a P_5, and consider the fives sets $R_1, \dots, R_5$ at the end of this procedure. By the definition of the procedure, it is easy to see that each R_i is complete to $R_{i+1} \cup R_{i-1}$ and anticomplete to $R_{i+2} \cup R_{i-2}$. By remark (a) in Section 3, we may assume that every vertex x in $V(G) \setminus R$ is located in the set Q_t that corresponds to the type t of x with respect to the collection $R_1, \dots, R_5$. Let W be the set of vertices of $V(G) \setminus R$ that have a neighbor in R. For each $i = 1, \dots, 5$, define the following sets:

- Let $W_i$be the set of all x in W such that x is anticomplete to R_i and has a neighbor in each R_j with $j \neq i$;
- Let $T_i$be the set of all x in W such that x is anticomplete to $R_{i-1} \cup R_{i+1}$ and has a neighbor in each R_j with $j \neq i \pm 1$.
- $Z = V(G) \setminus (R \cup W)$.

In other words, Z is the set of vertices of type $(0,0,0,0,0)$, W_1 is the set of vertices whose type is $(0,1,1,1,1)$, T_1 is the set of vertices whose type is $(1,0,1,1,0)$, and W_j, T_j are defined similarly when $j \in \{2, \dots, 5\}$. We say that $(R_1, \dots, R_5, W_1, \dots, W_5, T_1, \dots, T_5, Z)$ is a C_5*-structure* of G.

Lemma 7.1. *Either $W = W_1 \cup \dots \cup W_5 \cup T_1 \cup \dots \cup T_5$, or G contains a P_5 or a member of $\mathcal{F}$.*

Proof. Consider any w in W. If w is anticomplete to at least three of the sets $R_1, \dots, R_5$, then it is anticomplete to $R_j \cup R_{j+1}$ for some j. In that case w has been examined during the above procedure, and either the procedure has found a P_5 and stopped, or it has placed w in one of the R_i's; in either case there is a contradiction. So w is anticomplete to at most two of $R_1, \dots, R_5$; and, by the same argument, if it is anticomplete to two sets R_h, R_j then $j \neq h \pm 1$. Thus if w is anticomplete to two of $R_1, \dots, R_5$, then w is in T_i for some i, and if w is anticomplete to only one of $R_1, \dots, R_5$, then w is in W_i for some i. Finally, if

w is not anticomplete to any of $R_1, \ldots, R_5$, then there is a neighbor u_i of w in R_i for every i in $\{1, \ldots, 5\}$, and then $\{w, u_1, \ldots, u_5\}$ induces an F_2. $\square$

Lemma 7.2. *We may assume that each of $R_1, R_2, R_3, R_4, R_5, Z$ is a stable set.*

Proof. Suppose up to symmetry that either R_1 or Z contains two adjacent vertices u, v. Then $\{u, v, r_3, r_4\}$ induces a $2K_2$ for any r_3 in R_3 and r_4 in R_4. In that case, as mentioned at the beginning of this section, the algorithm returns the answer "G is not in class $\mathcal{C}$" and it is a correct answer. $\square$

For any two disjoint sets K and L, let:
K^L denote the set $\{x \in K \mid x \text{ has a neighbor in } L\}$,
K^{-L} denote the set $\{x \in K \mid x \text{ has a non-neighbor in } L\}$,
K^{0L} denote the set $\{x \in K \mid x \text{ has no neighbor in } L\}$.
(So $K^{0L} = K \setminus K^L$.) We will be interested in such sets for some pairs K, L among $R_1, \ldots, R_5, T_1, \ldots, T_5, W_1, \ldots, W_5, Z$. So their total number is a constant. By Remarks (a) and (b) of Section 3, all such sets can be computed in linear time. Let us know analyze the adjacency relation between the sets of the C_5-partition.

Lemma 7.3. *Either the following properties hold for each iin $\{1, \ldots, 5\}$, or G contains a P_5 or a member of $\mathcal{F}$.*

(1) T_i is complete to R_i and to $T_{i-1} \cup T_{i+1}$.
(2) T_i is a stable set.
(3) Every vertex of T_i is complete to R_{i-2} or to R_{i+2}.
(4) T_i^Z is complete to $R_{i-2} \cup R_{i+2}$.
(5) W_i is complete to $R_{i-1} \cup R_{i+1} \cup T_i \cup T_{i-2} \cup T_{i+2} \cup W_{i-1} \cup W_{i+1}$ and anticomplete to $T_{i-1} \cup T_{i+1}$.
(6) W_i is a stable set.
(7) If $W_i \neq \emptyset$, then $W_{i-2} \cup W_{i+2}$ is empty, T_i^Z is empty, and there is no edge between T_{i-1} and T_{i+1}.
(8) Some vertex of R_{i+3} is complete to $W_i \cup T_{i+1}$, and some vertex of R_{i-3} is complete to $W_i \cup T_{i-1}$.
(9) $T_{i-1}^{-R_{i+1}}$ is complete to $T_{i+1}^{-R_{i-1}}$.
(10) If $T_i^{0T_{i+2}} \neq \emptyset$, then every vertex of R_i is complete to T_{i+2} or to $T_{i+3}^{0T_i}$.
If $T_i^{0T_{i-2}} \neq \emptyset$, then every vertex of R_i is complete to T_{i-2} or to $T_{i-3}^{0T_i}$.
(11) Every vertex of Z is either complete or anticomplete to $T_{i-1}^{-T_{i+1}} \cup T_{i+1}^{-T_{i-1}}$.

Proof. For each item of the lemma, the proof is an algorithm that either finds an induced P_5 or a member of $\mathcal{F}$ or establishes the property. Throughout this proof, let r_j be an arbitrary vertex of R_j for each j in $\{1, \ldots, 5\}$.

(1) Up to symmetry, consider any vertex t_1 in T_1. By the definition of T_1, we know that t_1 has no neighbor in $R_2 \cup R_5$ and has neighbors $u_3$in R_3 and $u_4$in R_4. Suppose that t_1 has a non-neighbor u_1 in R_1. Then t_1-u_4-r_5-u_1-r_2 is a P_5. Now we may assume that T_j is complete to R_j for each j in $\{1, \dots, 5\}$. Suppose, up to symmetry, that t_1 has a non-neighbor $t_2$in T_2. Vertex t_2 has a neighbor v_5 in R_5. Then, u_3-t_1-r_1-v_5-t_2 is a P_5.

(2) Up to symmetry, suppose that T_1 contains two adjacent vertices u and v. By (1), u and v are adjacent to r_1. By the definition of T_1, u has neighbors u_3 in R_3 and u_4 in R_4. Likewise, v has neighbors v_3 in R_3 and v_4 in R_4. If $u_3 = v_3$ and $u_4 = v_4$, then $\{u, v, u_3, u_4\}$ induces a K_4. Now suppose, up to symmetry, that $u_3 \neq v_3$ and, by the same argument, that u is not adjacent to v_3 and v is not adjacent to u_3. If $u_4 = v_4$, then $\{u, v, r_1, r_2, u_3, v_3, u_4\}$ induces an F_6. So suppose that $u_4 \neq v_4$ and, by the same argument, that u is not adjacent to v_4 and v is not adjacent to u_4. Then u-r_1-r_2-v_3-v_4 is a P_5.

(3) Suppose that a vertex t_1 in T_1 has non-neighbors u_3 in R_3 and u_4 in R_4. We know that t_1 has a neighbor u_1 in R_1 and a non-neighbor u_2 in R_2. Then t_1-u_1-u_2-u_4-u_4 is a P_5.

(4) Let $t_1 \in T_1$ have a neighbor z in Z. Suppose that t_1 has a non-neighbor x in $R_3 \cup R_4$, say $x \in R_4$. We know that t_1 has a neighbor u_4 in R_4 and x has a neighbor u_5 in R_5. Then z-t_1-u_4-u_5-x is a P_5.

(5) Up to symmetry, consider any vertex w_1 in W_1. We know that w_1 has no neighbor in R_1 and has neighbors v_3 in R_3 and v_4 in R_4. First suppose that w has a non-neighbor x in $R_5 \cup R_2 \cup T_1 \cup T_4 \cup T_3 \cup W_5 \cup W_2$.
Suppose that $x \in R_2$. We know that w_1 has a neighbor u in R_2. Then x-r_1-u-w_1-u_4 is a P_5. The same holds if $x \in R_5$. Therefore we may assume that w_1 is complete to R_2 and to R_5, and that W_j is complete to $R_{j-1} \cup R_{j+1}$ for each j in $\{1, \dots, 5\}$.
Suppose that $x \in T_1$. If x is not adjacent to v_3, then x-r_1-r_5-w_1-v_3 is a P_5. So suppose that x is adjacent to v_3 and, similarly, to v_4. Then $\{w_1, x, r_1, r_2, v_3, v_4, r_5\}$ induces an F_6.
Suppose that $x \in T_3 \cup W_2$. Vertex x has a neighbor u_1 in R_1 and no neighbor in R_2. If x is not adjacent to v_4, then x-u_1-r_2-w_1-v_4 is a P_5. If x is adjacent to v_4, then $x \in W_2$, and so x is complete to $R_1 \cup R_3$ and has a neighbor v_5 in R_5. Then $\{w_1, x, r_1, r_2, v_3, v_4, u_5\}$ induces an F_8. The same holds if $x \in T_4 \cup W_5$.
Now suppose that w_1 has a neighbor $t_2$in T_2. Vertex t_2 has a neighbor u_5 in R_5. If t_2 is adjacent to v_4, then $\{w_1, t_2, v_4, u_5\}$ induces a K_4. If t_2 is not adjacent to v_4, then $\{w_1, t_2, r_2, v_3, v_4, u_5\}$ induces an F_2. The same holds for T_5.

(6) Suppose, up to symmetry, that W_1 contains two adjacent vertices u, v.

By (5), u and v are adjacent to r_2 and r_5 and not to r_1. By the definition of W_1, u has neighbors u_3 in R_3 and u_4 in R_4, and v has neighbors v_3 in R_3 and v_4 in R_4. If v is adjacent to u_3, then $\{u, v, r_2, u_3\}$ induces a K_4. So let $vu_3 \notin E$, and, similarly, $vu_4 \notin E$. Then $\{u, v, r_2, u_3, u_4, r_5\}$ induces an F_2.

(7) Up to symmetry, let w_1 be a vertex in W_1. We know that w_1 has neighbors v_3 in R_3 and v_4 in R_4.
First suppose that there is a vertex w_3 in W_3. Vertex w_3 has neighbors v_5 in R_5 and v_1 in R_1. If w_1 is adjacent to w_3, then, by (5), $\{w_1, w_3, v_4, v_5\}$ induces a K_4. If w_1 is not adjacent to w_3, then, by (5), $\{w_1, w_3, v_1, r_2, v_3, v_4, v_5\}$ induces an F_7. So $W_3 = \emptyset$, and similarly $W_4 = \emptyset$.
Now suppose that there is an edge t_1z with $t_1 \in T_1$ and $z \in Z$. By (3), t_1 is complete to $R_3 \cup R_4$. By (5), t_1 is adjacent to w_1. So $\{t_1, w_1, v_3, v_4\}$ induces a K_4.
Now suppose that there is an edge t_2t_5 with $t_2 \in T_2$ and $t_5 \in T_5$. Vertex t_2 has a neighbor u_5 in R_5, and t_5 has a neighbor u_2 in R_2. Then, by (1) and (5), $\{w_1, t_2, t_5, u_2, v_3, v_4, u_5\}$ induces an F_5, F_7 or F_9 (depending on the existence of edges t_2v_4 and t_5v_3).

(8) Up to symmetry, take $i = 1$. Let u be a vertex in R_4 with the largest number of neighbors in $W_1 \cup T_2$, and suppose that some vertex v in $W_1 \cup T_2$ is not adjacent to u. We know that v has a neighbor u_4 in R_4. By the choice of u, there is a vertex w in $W_1 \cup T_2$ that is adjacent to u and not to u_4. Then u-w-r_2-v-u_4 is a P_5.

(9) Up to symmetry, take $i = 2$. Suppose that some vertex t_1 in $T_1^{-R_3}$ is not adjacent to some vertex t_3 in $T_3^{-R_1}$. By the definition of $T_1^{-R_3}$, vertex t_1 has a non-neighbor x_3 in R_3, and similarly, t_3 has a non-neighbor x_1 in R_1. Then t_1-x_1-r_2-x_3-t_3 is an induced P_5.

(10) Take $i = 1$. Suppose that there exists a vertex t_1 in $T_1^{0T_3}$ and that some vertex x_1 in R_1 has a non-neighbor t_3 in T_3 and a non-neighbor t_4 in $T_4^{0T_1}$. By the definitions of these sets, we know that t_1 is not adjacent to t_3 and that t_4 is not adjacent to t_1. Vertex t_3 is adjacent to t_4 by (1) and to r_5 by (3). Then t_1-x_1-r_5-t_3-t_4 is an induced P_5.

(11) Take $i = 1$. Suppose that some vertex z in Z has a neighbor u and a non-neighbor v with u, v both in $T_5^{-T_2} \cup T_2^{-T_5}$. We may assume that $u \in T_5^{-T_2}$; so u has a non-neighbor u' in $T_2^{-T_5}$. If zu' is not an edge, then z-u-a_3-b_4-u' is an induced P_5. So let $zu' \in E(G)$. Now u, u' play symmetric roles, so we may assume that $v \in T_5^{-T_2}$. If vu' is not an edge, then z-u'-b_4-a_3-v is an induced P_5. So let $vu' \in E(G)$. Since v is in $T_5^{-T_2}$, it has a non-neighbor v' in T_2. If zv' is an edge, then z-v'-b_4-a_3-v is an induced P_5. So let $zv' \notin E(G)$. If uv' is an

edge, then v'-u-a_3-v-u' is an induced P_5. So let $uv' \notin E(G)$. Then z-u-a_3-b_4-v' is an induced P_5. □

7.1 Testing 3-colorability

Now we can test whether G is 3-colorable. For any mapping $\gamma : V(G) \to \{1, 2, \ldots\}$ and subset $X \subseteq V(G)$, let $\gamma(X)$ be the set $\{\gamma(x) \mid x \in X\}$.

Lemma 7.4. *If at least two of $R_1, \ldots, R_5$ contain a double vertex, or if W contains a double vertex, then G is not 3-colorable.*

Proof. First suppose, up to symmetry, that there is a double vertex u in R_1 and a double vertex v in R_j with $j \in \{2, 3\}$. Recall that there is a vertex $\overline{u}$ in G_0 that it adajcent to u and has the same neighbors as u in $V(G) \setminus \{u\}$, and similarly for v. If $j = 2$, then $\{u, \overline{u}, v, \overline{v}\}$ induces a K_4 in the original graph G_0. If $j = 3$, pick any r_i in R_i for each $i \in \{2, 4, 5\}$. Then $\{u, \overline{u}, v, \overline{v}, r_2, r_4, r_5\}$ induces an F_3 in G_0.

Now suppose that W contains a double vertex u. By the definition of W, there is an integer $i \in \{1, \ldots, 5\}$ such that u has neighbors u_i in R_i and u_{i+1} in R_{i+1}. Then $\{u, \overline{u}, u_i, u_{i+1}\}$ induces a K_4 in the original graph G_0. □

The hypotheses of the preceding lemma are easy to test in time $O(n)$. If any is satisfied, the algorithm stops with the answer "G is not 3-colorable." Let us now assume that W and at least four of $R_1, \ldots, R_5$ contain no double vertex.

Lemma 7.5. *If G admits a 3-coloring γ, then there is an integer h in $\{1, \ldots, 5\}$ such that (up to renaming colours), $\gamma(R_{h+1}) = \gamma(R_{h+4}) = 1$, $\gamma(R_{h+2}) = 2$ and $\gamma(R_{h+3}) = 3$.*

Proof. If all three colors appear in some R_i, then no color can appear in R_{i+1}, a contradiction. Therefore at most two colors appear in each R_i. Suppose, up to symmetry, that colors 1 and 2 appear in R_1. Then we must have $\gamma(R_2 \cup R_5) = 3$, so $\gamma(R_3 \cup R_4) \subseteq \{1, 2\}$, so $\gamma(R_3) = 1$ and $\gamma(R_4) = 2$ or vice-versa; thus the lemma holds with $h = 2$. Now suppose that only one color appears in each R_i. Then we may assume up to symmetry that $\gamma(R_1) = 1$, $\gamma(R_2) = 2$, $\gamma(R_3) = 3$, $\gamma(R_4) = 1$, $\gamma(R_5) = 2$, and the lemma holds with $h = 5$. □

For each h in $\{1, \ldots, 5\}$, let γ_h be the *precoloring* (partial coloring) where color 1 is assigned to the vertices of $R_{h+1} \cup R_{h+4}$, color 2 to the vertices of R_{h+2}, color 3 to the vertices of R_{h+3}, and the other vertices are uncolored. The preceding lemma shows that, in order to test whether G is 3-colorable, it suffices to test, for each h in $\{1, \ldots, 5\}$, whether γ_h extends to a 3-coloring of G. This can be done easily, and in linear time, as follows.

Initially, every vertex x has a list of available colors $L(x) = \{1, 2, 3\}$. Every vertex of $R \setminus R_h$ receives the color assigned by γ_h; in particular, if $R \setminus R_h$ contains a double vertex, then the algorithm declares that γ_h does not extend to a 3-coloring (the algorithm records the uncolorable vertex, and if $h < 5$, the algorithm examines γ_{h+1}, else the algorithm stops with the answer "G is not 3-colorable"). When a vertex is colored, its color is removed from the list of its neighbors. If the list of a vertex has size one, or if the list of a double vertex has size two, then we color that vertex with the available color(s) and update the list of its neighbors. If the list of a vertex becomes empty, or if the list of a double vertex has size one, then that vertex is uncolorable and the algorithm declares that γ_h does not extend to a 3-coloring (again, the algorithm records the uncolorable vertex, and if $h < 5$, the algorithm examines γ_{h+1}, else the algorithm stops with the answer "G is not 3-colorable"). A crucial observation is that *every vertex of W has two neighbors of different colors in $R \setminus R_h$*. This follows directly from the definition of W. This observation means that, just after the vertices of $R \setminus R_h$ are colored, every vertex of W has a list of size at most one, and this ensures that either the algorithm stops because some vertex in W is uncolorable or all vertices of W can be colored. Moreover, we may assume that all vertices of W are colored before the vertices of $R_h \cup Z$ are considered. Finally, the remaining uncolored vertices are in $R_h \cup Z$, which is a stable set, so we may examine them one by one independently of each other. If any such vertex x is uncolorable, then the algorithm records its (and if $h < 5$, the algorithm examines γ_{h+1}), else the algorithm assigns to x a color from $L(x)$ (or two colors if x is a double vertex).

Throughout this procedure, coloring a vertex u and updating its neighbors takes time $O(d(u))$, so the total running time is $O(n + m)$.

At the end of the procedure, there are two possible outcomes:
(1) The algorithm tried each γ_h and none of them extends to a 3-coloring. The algorithm returns the correct answer "G is not 3-colorable" and stops. In Section 7.2 we show that in that case G contains a member of $\mathcal{F}$.
(2) Some γ_h extends to a 3-coloring. In Section 7.3 we analyse this situation in order to find a stable set of maximum weight.

7.2 When G is not 3-colorable

For two sets $X, Y \subset V(G)$, let us call *XY-edge* any edge xy with $x \in X$ and $y \in Y$.

Lemma 7.6. *Suppose that γ_h does not extend to a 3-coloring of G. Then either G_0 contains a member of $\mathcal{F}$ or one of the following holds:*
(i) $R \setminus R_h$ contains a double vertex.
(ii) $W_{h-1} \cup W_h \cup W_{h+1} \neq \emptyset$.
(iii) There is a $T_{h+1}T_{h+3}$-edge or a $T_{h-1}T_{h-3}$-edge.
(iv) R_h contains a vertex that has a neighbor in $T_{h+2} \cup W_{h+3}$ and a neighbor in $T_{h-2} \cup W_{h-3}$.
(v) A vertex of Z has a neighbor in each of T_h, T_{h-2}, T_{h+2}.
(vi) A double vertex of Z either has a neighbor in T_h and a neighbor in $T_{h-2} \cup T_{h+2}$ or has a neighbor in T_{h-1} and a neighbor in T_{h+1}.

Proof. To simplify the notation, let us put $h = 5$. So the vertices of $R_1 \cup R_4$ have color 1, the vertices of R_2 have color 2, and the vertices of R_3 have color 3. Suppose that none of (i)–(iii) holds. After the algorithm has colored the vertices of $R \setminus R_5$, the definition of the T_i's and W_i's implies that every vertex x in T_1 satisfies $L(x) = \{2\}$, because x has a neighbor in each of R_1, R_3, R_4; we denote this fact by $L(T_1) = \{2\}$. Likewise, we find that $L(T_2) = \{3\}$, $L(T_3) = \{2\}$, $L(T_4) = \{3\}$, $L(T_5) = \{1\}$, $L(W_2) = \{2\}$ and $L(W_3) = \{3\}$. Note that, since (i) does not hold and by Lemma 7.4, there is no double vertex in the set $(R \setminus R_5) \cup W$. Since (ii) and (iii) do not hold, and by Lemma 7.3 (5), there is no color conflict between the vertices of W. Thus γ_5 extends to a 3-coloring of the subgraph induced by $(R \setminus R_5) \cup W$. So the reason for which γ_5 does not extend to a coloring of G is that there exists a vertex x in $R_5 \cup Z$ that is uncolorable. This means that, after the vertices of W have been colored, we have one of the following two cases. For each $i \in \{1, \ldots, 5\}$, pick any r_i in R_i.

Case 1: $L(x) = \emptyset$.
So x has neighbors u, v, w of color 1, 2 and 3 respectively. Since x is in $R_5 \cup Z$, it has no neighbor in $R_2 \cup R_3$, therefore $v \in T_1 \cup T_3 \cup W_2$ and $w \in T_4 \cup T_2 \cup W_3$. First let $x \in R_5$. By the definition of the T_i's and W_i's, we have $v \in T_3 \cup W_2$ and $w \in T_2 \cup W_3$. Thus we get outcome (iv) of the lemma. Now let $x \in Z$. So $u \in T_5$. By Lemma 7.3 (3), u is complete to $R_2 \cup R_3$. According to the position of v and w, there are nine subcases:
Case 1.1: $v \in T_1$ and $w \in T_4$. By Lemma 7.3 (1), u is adjacent to v and w. If v is adjacent to w, then $\{u, v, w, x\}$ induces a K_4. If v is not adjacent to w, then

$\{u, v, w, x, r_2, r_3\}$ induces an F_2.
Case 1.2: $v \in T_1$ and $w \in T_2$. This case is similar to Case 1.1 (u, v, w are in three consecutive T_i's).
Case 1.3: $v \in T_1$ and $w \in W_3$. By Lemma 7.3 (1), u is adjacent to v, and by (5) w is adjacent to u and v. Then $\{u, v, w, x\}$ induces a K_4.
Case 1.4: $v \in T_3$ and $w \in T_4$. This case is similar to Case 1.1.
Case 1.5: $v \in T_3$ and $w \in T_2$. Then we get outcome (v) of this lemma.
Case 1.6: $v \in T_3$ and $w \in W_3$. Then Lemma 7.3 (3) is contradicted.
Case 1.7: $v \in W_2$ and $w \in T_4$. This case is similar to Case 1.3.
Case 1.8: $v \in W_2$ and $w \in T_2$. Then Lemma 7.3 (3) is contradicted.
Case 1.9: $v \in W_2$ and $w \in W_3$. By Lemma 7.3 (5), u, v and w are pairwise adjacent. So $\{u, v, w, x\}$ induces a K_4.

Case 2: x is a double vertex with $|L(x)| \leq 1$.
First let $x \in R_5$. Then x must have a neighbor u of color 2 or 3, and we may assume, up to symmetry, that u is in $W_2 \cup T_3$. So u has a neighbor u_1 in R_1. Then $\{x, \overline{x}, u, u_1\}$ induces a K_4 in G_0. Now let $x \in Z$. So x has two neighbors u, v of different colors, and we have $u, v \in T_1 \cup \cdots \cup T_5 \cup W_2 \cup W_3$. If u and v are adjacent, then $\{x, \overline{x}, u, v\}$ induces a K_4. So assume that u and v are not adjacent. By Lemma 7.3 (1) and (5), this yields the following subcases.
Case 2.1: $u \in T_5$ and $v \in T_2 \cup T_3$. Then we get outcome (vi) of this lemma.
Case 2.2: $u \in T_1$ and $v \in T_4$. Then we get outcome (vi) of this lemma.
Case 2.3: $u \in T_j$ and $v \in W_j$ for $j \in \{2, 3\}$. Then Lemma 7.3 (3) is contradicted. This completes the proof of Lemma 7.6. □

Lemma 7.7. *If the algorithm declares that G is not 3-colorable, then G contains a member of $\mathcal{F}$.*

Proof. We first observe that when the algorithm declares that G is not 3-colorable, this is a correct answer, because of the arguments presented in Section 7.1. Now, since G is not 3-colorable, Theorem 1.4 and Lemma 1.5 imply that G contains a member of $\mathcal{F}$. Let us now show how the algorithm can actually find such a subgraph in linear time.

For each h in $\{1, \ldots, 5\}$, the algorithm finds that the precoloring γ_h does not extend to a 3-coloring. So each γ_h satisfies one of the outcomes (i)—(vi) of Lemma 7.6, and the algorithm finds a set A_h of vertices that cause such an outcome. In other words and more precisely: if (i) of Lemma 7.6 holds, A_h consists of a double vertex from $R \setminus R_h$; if (ii) holds, A_h consists of a vertex w from $W_{h-1} \cup W_h \cup W_{h+1}$; if (iii) holds, A_h consists of adjacent vertices t, t' with either $t \in T_{h+1}$ and $t' \in T_{h+3}$ or $t \in T_{h-1}$ and $t' \in T_{h-3}$-edge; if (iv) holds, A_h consists of three vertices x, t, t' such that $x \in R_h$, $t \in T_{h+2} \cup W_{h+3}$, $t' \in T_{h-2} \cup W_{h-3}$ and x is adjacent to t and t'; if (v) holds, A_h consists of a vertex z in Z and three neighbors t, t', t'' of z with $t \in T_h$, $t' \in T_{h-2}$, $t'' \in T_{h+2}$; and if (vi) holds, A_h, consists of a double vertex z of Z and two neighbors t, t' of z with either $t \in T_h$ and $t' \in T_{h-2} \cup T_{h+2}$ or $t \in T_{h-1}$ and $t' \in T_{h+1}$. Note that in any case we have $|A_h| \leq 4$. By Lemma 7.3 (8), for each i the set R_i contains a vertex a_i that is complete to $T_{i+2} \cup W_{i+3}$ and a vertex b_i that is complete to $T_{i-2} \cup W_{i-3}$ (possibly $a_i = b_i$). Let H be the subgraph of G that is induced by the subset $\{a_1, b_1, \ldots, a_5, b_5\} \cup A_1 \cup \cdots \cup A_5$. Then we know that H itself is not 3-colorable, since each γ_h satisfies an item of Lemma 7.6 in H. So H contains a member of $\mathcal{F}$. Note that H has at most 30 vertices, so finding a member of $\mathcal{F}$ in H can be done in constant time.

We now analyse this situation, which will give a direct proof (not using Theorem 1.4) of Theorem 1.1. Throughout this proof, let r_i be an arbitrary vertex of R_i for each $i \in \{1, \ldots, 5\}$; for example r_i can be a_i or b_i.

Case 1: R contains a double vertex.
We may assume up to symmetry that R_5 contains a double vertex x and, by Lemma 7.4, that $R \setminus R_5$ does not contain a double vertex. So γ_5 does not satisfy (i) of Lemma 7.6.
Suppose that γ_5 satisfies (ii); so there is a vertex $w \in W_4 \cup W_5 \cup W_1$. If w is in W_5, then $\{x, \overline{x}, w, r_1, a_2, b_3, r_4\}$ induces an F_5 in G_0. If $w \in W_4 \cup W_1$, then it is adjacent to x and it has a neighbor u in $R_1 \cup R_4$, so $\{x, \overline{x}, w, u\}$ induces a K_4.
Suppose that γ_5 satisfies (iii); so, up to symmetry, there is an edge t_1t_3 with $t_1 \in T_1$ and $t_3 \in T_3$. We know that t_1b_3, t_1a_4 and t_3a_1 are edges. If t_3 is adjacent to x, then $\{x, \overline{x}, a_1, t_3\}$ induces a K_4. If t_3 is not adjacent to x, then $\{x, \overline{x}, a_1, b_3, a_4, t_1, t_3\}$ induces an F_5.
Suppose that γ_5 satisfies (iv); so some vertex u in R_5 has neighbors v in $T_2 \cup W_3$ and w in $T_3 \cup W_2$. We know that vr_2, vb_4, wr_3, wa_1 are edges and vr_3, wr_2 are not edges. By Lemma 7.3 (5), v and w are adjacent. If v is adjacent to x, then $\{x, \overline{x}, b_4, v\}$ induces a K_4. If v is adjacent to a_1, then $\{u, v, w, a_1\}$ induces a K_4. So assume that v is not adjacent to x or to a_1, and, similarly, that w is not adjacent to x or b_4. Then $\{x, \overline{x}, u, v, w, b_4, a_1\}$ induces an F_5.
Suppose that γ_5 satisfies (v); so some vertex z in Z has a neighbor t_2 in T_2. By 7.3 (3), t_2 is complete to $R_4 \cup R_5$. Then $\{x, \overline{x}, t_2, r_4\}$ induces a K_4.
Suppose that γ_5 satisfies (vi) for some double vertex $z \in Z$. If z has a neighbor in T_2, then we can argue as when γ_5 satisfies (v) above. So assume that z has neighbors t_1 in T_1 and t_4 in T_4. By 7.3 (3), t_1 is complete to $R_3 \cup R_4$, and t_4 is complete to $R_1 \cup R_2$. If t_1 and t_4 are adjacent, then $\{z, \overline{z}, t_1, t_4\}$ induces a K_4.

If t_1 and t_4 are not adjacent, then $\{x, \overline{x}, z, \overline{z}, t_1, t_4, r_1, r_2, r_3, r_4\}$ induces an F_{10}. This completes the proof of case 1.

Now suppose that some γ_h satisfies (vi). So there is a double vertex z in Z and an integer i in $\{1, \dots, 5\}$ such that z has a neighbor t_i in T_i and a neighbor t_{i+2} in T_{i+2}. If t_i and t_{i+2} are adjacent, then $\{z, \overline{z}, t_i, t_{i+2}\}$ induces a K_4. If t_i and t_{i+2} are not adjacent, then, by Lemma 7.3 (3), $\{z, t_i, r_{i+3}, r_{i+4}, t_{i+2}\}$ induces a C_5. We can then restart the algorithm from this C_5, and z will be a double vertex in the new set R, so we can conclude as in Case 1. Therefore we may assume that no γ_h satisfies (vi).

Case 2: Two of the sets $W_1, \dots, W_5$ are not empty.
By Lemma 7.3 (7) and up to symmetry, we may assume that there exist vertices w_2 in W_2 and w_3 in W_3 and that $W_1 \cup W_4 \cup W_5 = \emptyset$. Thus γ_5 does not satisfy (ii). By the preceding points, we know that γ_5 does not satisfy (i) or (vi). By Lemma 7.3 (7), there is no edge between T_1 and T_3, or between T_2 and T_4, or between $T_2 \cup T_3$ and Z, so γ_5 does not satisfy (iii) or (v). Hence γ_5 satisfies (iv), that is, some vertex a in R_5 has a neighbor u in $W_2 \cup T_3$ and a neighbor v in $W_3 \cup T_2$. (Possibly $u = w_2$ or $v = w_3$.) By Lemma 7.3 (1) and (5), each of u, w_2 is adjacent to each of v, w_3. If $u \in W_2$ and $v \in W_3$, then $\{a_4, a, u, v\}$ induces a K_4. Therefore assume, up to symmetry, that $u \in T_3$; so $u \neq w_2$. We know that w_2a_4, w_3b_1, w_2b_5, ub_5, w_3a_5, va_5 and ua_1 are edges. We may assume that b_5 is not adjacent to w_3 (else take b_5, w_2, w_3 instead of a, u, v) and similarly a_5 is not adjacent to w_2. So $b_5 \neq a_5$.
Suppose that $v \in W_3$. If a is adjacent to w_2, then $\{a, v, w_2, a_4\}$ induces a K_4. So let $aw_2 \notin E$. Then $a \neq b_5$ and, by the same argument, v is not adjacent to b_5. If va_1 is an edge, then $\{v, a_1, a, u\}$ induces a K_4; else $\{a, u, v, a_1, w_2, a_4, b_5\}$ induces an F_7.
Now suppose that $v \in T_2$; so $v \neq w_3$ and the symmetry between u and v is restored. Then a is not adjacent to w_2 (else take $u = w_2$), and similarly a is not adjacent to w_3. If u is adjacent to b_1 and v is adjacent to a_4, then $\{a, u, v, w_2, w_3, b_1, a_4\}$ induces an F_9 ($= \overline{C}_7$). So suppose, up to symmetry, that $ub_1 \notin E$ and, by the same argument, that $w_3a_1 \notin E$. Then $\{u, a_1, b_1, w_2, w_3, b_5, a_5\}$ induces an F_9.

Case 3: Exactly one of $W_1, \dots, W_5$ is not empty.
Up to symmetry, assume that W_1 is not empty and $W_2 \cup W_3 \cup W_4 \cup W_5 = \emptyset$. Thus γ_3 and γ_4 do not satisfy (ii). By the preceding points, γ_3 and γ_4 do not satisfy (i) or (vi). By Lemma 7.3 (7), there is no edge between T_1 and Z, so γ_3 and γ_4 does not satisfy (v). Hence γ_3 and γ_4 satisfy (iii) or (iv). Note also

that, by Lemma 7.3 (7), there is no edge between T_2 and T_5. Therefore: for γ_3 either there is a T_1T_4-edge or some vertex in R_3 has a neighbor in $T_5 \cup W_1$ and a neighbor in $T_1 \cup W_5$; and, for γ_4, either there is a T_1T_3-edge or some vertex in R_4 that has a neighbor in $T_1 \cup W_2$ and a neighbor in $T_2 \cup W_1$. Up to symmetry, this leads to the two subcases below. First we pick a vertex w_1 in W_1. We know that w_1 is complete to $R_2 \cup R_5$, anticomplete to R_1 and adjacent to a_3 and b_4.

Case 3.1: There is (for γ_3) an edge t_1t_4 with $t_1 \in T_1$ and $t_4 \in T_4$.
By Lemma 7.3, we know that w_1t_1, w_1t_4, t_1b_3, t_1a_4, t_4a_4, t_4b_4 are edges, and that t_4b_3, t_4a_3 are not edges. If t_1b_4 is an edge, then $\{t_1, b_4, w_1, t_4\}$ induces a K_4. If w_1a_4 is an edge, then $\{w_1, a_4, t_1, t_4\}$ induces a K_4. So assume that t_1b_4 and w_1a_4 are not edges. Thus $a_4 \neq b_4$.
Suppose that (for γ_4) there is an edge s_1s_3 with $s_1 \in T_1$ and $s_3 \in T_3$. (Possibly $s_1 = t_1$.) As above, we know that w_1s_1, w_1s_3, s_1b_3, s_1a_4, s_3b_3 are edges, that s_3a_4, s_3b_4, w_1b_3 are not edges, and that $a_3 \neq b_3$. By Lemma 7.3 (1), s_3t_4 is an edge and s_1t_1 is not an edge. If t_1s_3 is an edge, then $\{t_1, s_3, w_1, t_4\}$ induces a K_4. Hence let $t_1s_3 \notin E$ and, similarly, $s_1t_4 \notin E$. Then $t_1 \neq s_1$, and $\{w_1, s_1, t_1, s_3, t_4, b_3, a_4\}$ induces an F_9.
Now suppose that (for γ_4) some vertex b in R_4 has a neighbor v in $W_1 \cup T_2$ and a neighbor s_1 in T_1. (Possibly $v = w_1$, or $s_1 = t_1$, or $b \in \{a_4, b_4\}$.) By Lemma 7.3 (8), we know that t_4a_2, w_1a_2, w_1s_1, vt_1, vs_1, va_2, vb_4, s_1b_3, s_1a_4, t_4b are edges and that s_1t_1, s_1a_2, t_1a_2, vw_1 are not edges.
Suppose that bw_1 is an edge. So $b \neq a_4$. If bt_1 is an edge, then $\{b, t_1, w_1, t_4\}$ induces a K_4. If bt_1 is not an edge, then $t_1 \neq s_1$, and the set $\{w_1, t_1, s_1, t_4, b, b_3, a_4\}$ hosts an F_9 ($= \Gamma_7$), so, by Lemma 1.5, G contains an induced F_9 or F_1.
Now assume that bw_1 is not an edge. So $b \neq b_4$, $v \neq w_1$, and we may assume that $v \in T_2$ (else take $w_1 = v$). We know that va_5 and w_1a_5 are edges and va_3 is not an edge. If bt_1 is not an edge, then $s_1 \neq t_1$, $b \neq a_4$, and the set $\{w_1, a_4, b, b_4, t_1, s_1, a_5, t_4, b_3, v\}$ hosts an F_{11} (with optional edges t_4s_1, vt_4, s_1b_4, b_3w_1, va_4), so, by Lemma 1.5, G contains an induced F_{11}, F_9 or F_1. Hence let bt_1 be an edge. If t_1a_3 is an edge, then $\{w_1, b, b_4, t_1, a_5, a_3, v\}$ induces an F_9. If t_1a_3 is not an edge and w_1b_3 is an edge, then $\{w_1, b, b_4, t_1, a_5, b_3, v\}$ induces an F_9. If t_1a_3 and w_1b_3 are not edges, then $\{v, t_4, a_5, t_1, b_4, b, w_1\}$ hosts an F_9 (with optional edge vt_4).

Case 3.2: Some vertex a in R_3 has a neighbor u in $W_1 \cup T_5$ and a neighbor t_1 in T_1, and some vertex b in R_4 has a neighbor v in $W_1 \cup T_2$ and a neighbor s_1 in T_1.
By Lemma 7.3 (1), (5) and (8), we know that $\{b_2, a_5, t_1, s_1\}$ is a stable set and is complete to $\{r_1, u, v, w_1\}$, and that u, v are not adjacent to r_1, w_1. Moreover,

t_1 and s_1 are adjacent to b_3 and a_4, and ua_3 and vb_4 are edges.
Suppose that aw_1 and bw_1 are edges. If bt_1 is an edge, then $\{b, t_1, u, a\}$ induces a K_4. So let $bt_1 \notin E$, and similarly $as_1 \notin E$. Then $a \neq b_3$, $b \neq a_4$, and $\{u, a, b, t_1, s_1, b_3, a_4\}$ hosts an F_9.
Now suppose that bw_1 is an edge and aw_1 is not. Thus $w_1 \neq u$ and $a_3 \neq a$. If s_1a_3 is an edge, then $\{s_1, a_3, b, w_1\}$ induces a K_4. So let $s_1a_3 \notin E$. Thus $b_3 \neq a_3$. If we can choose $s_1 = t_1$, then $\{s_1, a_3, a, w_1, u, b, b_2\}$ induces an F_9. So let $s_1 \neq t_1$, and, by the same argument, $as_1, bt_1 \notin E$. Thus $b_3 \neq a$ and $a_4 \neq b$. If ub_3 is an edge, then $\{s_1, a_3, b_3, w_1, u, b, b_2\}$ hosts an F_9 (with optional edge b_3w_1). So let $ub_3 \notin E$. If w_1a_4 is an edge, then $\{w_1, u, a_4, b_2, t_1, a_3, a\}$ hosts an F_9 (with optional edges ua_4, t_1a_3). So let $w_1a_4 \notin E$. Thus $a_4 \notin \{b, b_4\}$. Then $\{s_1, a_3, a, b_3, w_1, u, a_4, b, b_4, t_1\}$ hosts an F_{11} (with optional edges t_1a_3, t_1b_4, s_1b_4).
Therefore we may assume that aw_1 and bw_1 are not edges, and more generally that a and b have no neighbor in W_1. Thus we have $u \in T_5$, $v \in T_2$, and $w_1 \notin \{u, v\}$. Also $a_3 \neq a$ and $b_4 \neq b$. Since $v \in T_2$, we have $va, va_3 \notin E$; and similarly, $ub, ub_4 \notin E$. If we can choose $s_1 = t_1$, then $\{s_1, a_5, a_3, a, v, w_1, u, b, b_4, b_2\}$ hosts an F_{11} (with optional edges uv, s_1a_3, s_1b_4) so G contains an induced F_1, F_9 or F_{11}. Therefore let $s_1 \neq t_1$ and, by the same argument, let $as_1, bt_1 \notin E$. So $b_3 \neq a$ and $a_4 \neq b$. We know that $vb_3, ua_4 \notin E$. If t_1b_4 is an edge, then $\{t_1, a_3, a, w_1, u, b_4, b_2\}$ hosts an F_9 (with optional edge t_1a_3). So let $t_1b_4 \notin E$, and similarly $s_1a_3 \notin E$. Thus $b_3 \neq a_3$ and $a_4 \neq b_4$. Then $\{a, b_3, v, w_1, u, a_4, b, b_4, b_2, t_1, s_1, a_5, a_3\}$ hosts an F_{12} (with optional edges uv, $w_1b_3, w_1a_4, t_1a_3, s_1b_4, ub_3, va_4$), so, by Lemma 1.5, G contains an induced F_{12}, F_{11}, F_9 or F_1. This completes the proof of case 3.

We may now assume that each of $W_1, \ldots, W_5$ is empty. Let J be the graph with six vertices $x_0, x_1, \ldots, x_5$ such that $x_1, \ldots, x_5$ induce a C_5 and x_0 is adjacent to exactly four of them. If G contains a copy of J, then we can restart the procedure at the beginning of this section with the five vertices $x_1, \ldots, x_5$. Then vertex x_0 is placed in one of the corresponding sets W_i, and we can conclude as in Cases 2 or 3 above. Therefore we may assume that G contains no J. Moreover, the following property (T) holds:

(T) *For any edge xy with $x \in T_{i-1}$ and $y \in T_{i+1}$, x is complete to R_{i+2} and y is complete to R_{i-2}.*
Indeed, if x has a non-neighbor z in T_{i+2}, then $\{a_{i-1}, b_{i+1}, b_{i-2}, x, y, z\}$ induces a J, and the same holds for y be symmetry.

Case 4: Some γ_h satisfies (v).
Let γ_5 satisfy (v). So some vertex z in Z has neighbors t_2 in T_2, t_3 in T_3 and

t_5 in T_5. By Lemma 7.3 (1), we have $t_2t_3 \in E$, and 7.3 (4) applies to t_2, t_3, t_5. If both t_5t_2 and t_5t_3 are edges, then $\{t_5, t_2, t_3, z\}$ induces a K_4. If only one is an edge, say $t_5t_2 \in E$ and $t_5t_3 \notin E$, then $\{r_1, r_2, z, t_5, t_2, t_3\}$ induces a J. So assume $t_5t_2, t_5t_3 \notin E$. Consider any t_1 in T_1. By Lemma 7.3 (1) and (8), we know that t_1t_2, t_1t_5 t_1b_3, t_1a_4 are edges. Then $\{b_3, a_4, t_2, z, t_5, t_1\}$ induces an F_2 or a J. Therefore $T_1 = \emptyset$, and similarly $T_4 = \emptyset$. Thus γ_3 does not satisfy (iv) or (v). By the preceding points, γ_3 does not satisfy (i), (ii) and (vi). So γ_3 satisfies (iii) and so (since $T_1 \cup T_4 = \emptyset$) there is an edge s_5s_2 with $s_5 \in T_5$ and $s_2 \in T_2$. Suppose that s_5 is adjacent to z. Then (just like for t_5) s_5 is complete to $R_2 \cup R_3$, s_5t_2 and s_5t_3 are not edges, and s_2 is not adjacent to z. Then $\{r_1, r_2, s_5, z, t_3, s_2\}$ induces a J. Now assume that s_5 is not adjacent to z. So $s_5 \neq t_5$. We know that s_5b_2 and s_5a_3 are edges. Then s_5 is adjacent to t_2, for otherwise z, t_2, s_5 violate Lemma 7.3 (11), and similarly s_5 is adjacent to t_3. But then $\{s_5, t_2, t_3, u_5\}$ induces a K_4. This completes the proof of case 4.

Therefore, each γ_h satisfies (iii) or (iv).

Case 5: For some i, there is a T_iT_{i+2}-edge and a T_iT_{i-2}-edge.
Assume that there are edges t_5t_2 and t'_5t_3 with $t_5, t'_5 \in T_5$, $t_2 \in T_2$, and $t_3 \in T_3$. We know that t_5 and t'_5 are adjacent to b_2 and a_3 and that t_2t_3, t_2b_4, t_2a_5, t_3a_1, t_3b_5 are edges. We may assume that either $t'_5 = t_5$ or t'_5t_2 and t_5t_3 are not edges. If some vertex x in R_5 is adjacent to both t_2 and t_3, then either $\{x, t_2, t_3, t_5\}$ induces a K_4 (if $t'_5 = t_5$) or $\{x, b_2, t_3, t_5, t'_5, t_2, a_3\}$ induces an F_9 (else). So we may assume that there is no such x. Thus $a_5t_3 \notin E$ and $b_5t_2 \notin E$.
Suppose that γ_5 satisfies (iii). So, up to symmetry, there is an edge $t_1t'_3$ with $t_1 \in T_1$ and $t'_3 \in T_3$. We have $t_1b_3 \in E$. We can argue with the pair of edges $\{t_1t'_3, t'_5t_3\}$ as above with $\{t_5t_2, t'_5t_3\}$, so we know that t_1a_3 and t'_5b_3 are not edges. By Property (T), t'_3 is complete to R_5, so $t'_3 \neq t_3$. Likewise, t_5 is complete to R_3, so $t_5 \neq t'_5$. Then $\{a_1, t_2, a_3, b_5, t_1, b_2, t_3, t'_3, t_5, t'_5\}$ induces an F_{11}.
Suppose that γ_5 satisfies (iv). So there is a vertex $a \in R_5$ that has neighbors s_2 in T_2 and s_3 in T_3. By the argument about x above, and up to symmetry, we may assume that $at_2 \notin E$. So $s_2 \neq t_2$, and we may assume that $s_2t_5, s_2t'_5 \notin E$ (else we could take $t_2 = s_2$). If s_2b_5 is an edge, then either $t'_5 = t_5$ and $\{b_5, a_5, t_3, b_4, t_5, s_2, t_2\}$ induces an F_9, or $t'_5 \neq t_5$ and $\{b_5, a_5, b_2, t_3, b_4, t_5, t'_5, s_2, t_2, a_3\}$ induces an F_{11}. So let $s_2b_5 \notin E$. By the same argument, with a instead of b_5, we obtain that $at_3 \notin E$. So $s_3 \neq t_3$, and as above we may assume that $s_3t_5, s_3t'_5 \notin E$ and $s_3a_5 \notin E$. Then $\{a, a_5, v_2, t_3, s_3, b_4, t'_5, t_5, a_1, s_2, t_2, a_3, b_5\}$ induces an F_{12}.

Case 6: For some i, there is a T_iT_{i+2}-edge and a $T_{i+1}T_{i+3}$-edge.
Assume that there are edges t_1t_3 and t_2t_4 with $t_i \in T_i$ for each $i = 1, 2, 3, 4$. We

know that t_1t_2, t_2t_3, t_3t_4, t_1b_3, t_2b_4, t_3a_1, t_4a_2 are edges and, by Property (T), t_1b_4, t_4a_1, t_2r_5 and t_3r_5 are edges. We may assume that we are not in Case 5, so there is no edge between T_1 and T_4 or between $T_2 \cup T_3$ and T_5. Therefore γ_2 and γ_3 satisfy (iv), so there is a vertex t_5 in T_5. By Lemma 7.3 (3) and up to symmetry, we may assume that t_5 is complete to R_2. We know that t_5a_3 is an edge. If t_5b_3 is an edge, then $\{t_5, t_3, a_2, t_1, t_4, b_3, t_2\}$ induces an F_9. Hence let $t_5b_3 \notin E$; so $b_3 \neq a_3$ and, by the same argument, $t_1a_3 \notin E$. If a_3 has a neighbor t'_1 in T_1, then we may assume that $t'_1t_3 \notin E$ (else take $t_1 = t'_1$), and then $\{t_1, t'_1, a_2, t_3, b_4, t_5, a_1, b_3, a_3, r_5\}$ induces an F_{11}. Hence let a_3 be anticomplete to T_1. Suppose that b_3 has a neighbor t'_5 in T_5. We may assume that $t'_5a_2 \notin E$ (else take $t_5 = t'_5$), so, by Lemma 7.3 (3), t'_5 is complete to R_3, and we know that t_5 and t'_5 are adjacent to b_2. Then $\{b_2, a_2, b_4, t'_5, t_5, t_2, b_3, a_3, t_4, t_1\}$ hosts an F_{11} (with optional edge t_4b_2). Hence let b_3 be anticomplete to T_5. Since γ_3 satisfies (iv), there is a vertex y in R_3 with a neighbor s_1 in T_1 and a neighbor s_5 in T_5. If yt_1 is an edge, then we can conclude as above with y instead of b_3. So let $yt_1 \notin E$ and, by the same argument, $s_1t_3 \notin E$. Since a_3 is anticomplete to T_1, we have $a_3s_1 \notin E$, so, by Lemma 7.3 (3), s_1 is complete to R_4. Likewise, $b_3s_5 \notin E$, so $s_5a_2 \in E$. Then $\{s_1, a_2, t_3, b_4, s_5, a_1, b_3, y, t_4, t_1\}$ induces an F_{11}. This concludes the proof of Case 6.

We may now assume that we are not in cases 1—6. So there is a T_iT_{i+2}-edge for at most one value of i, and we may assume up to symmetry that if there is such an edge then $i = 4$. Therefore γ_1, γ_4 and γ_5 do not satisfy (iii), so they satisfy (iv), that is, for each $j \in \{1, 4, 5\}$ there is a vertex u_j in R_j that has neighbors t_{j+2} in T_{j+2} and s_{j+3} in T_{j+3}. Also we know that T_1 is complete to T_2 and anticomplete to T_3, that T_4 is complete to T_3 and anticomplete to T_2, and that T_2 is complete to T_3.

Case 7: There is a T_4T_1-edge.
Let t_4s_1 be an edge with $t_4 \in T_4$ and $s_1 \in T_1$. We choose vertices t_i and s_i ($i \in \{1, 2, 3, 4\}$) such that $t_i = s_i$ if possible, i.e., whenever one vertex of T_i can play the role of both t_i and s_i. (Thus, if, for example, $t_2 \neq s_2$, then u_4 is not adjacent to t_2 and u_5 is not adjacent to s_2. This also implies $u_4 \neq a_4$ and $u_5 \neq a_5$.) Consider the set $D = \{i \in \{1, 2, 3, 4\} \mid t_i \neq s_i\}$.
If $D = \emptyset$, then $\{u_1, u_4, t_3, t_1, u_5, t_4, t_2\}$ induces an F_9.
Now suppose that D contains 1. Then s_1 is not adjacent to u_4, t_1 has no neighbor in T_4 and $u_4 \neq a_4$. If we also have $3 \in D$, then s_3 is not adjacent to u_1, so we have $s_3 \in T_3^{-R_1}$ and $t_1 \in T_1^{-T_4}$, and these two vertices violate Lemma 7.3 (10). Hence $3 \notin D$, that is, $t_3 = s_3$. Suppose that $4 \in D$. Then t_4 is not adjacent to u_1, s_4 has no neighbor in T_1, and $u_1 \neq b_1$. Moreover, by the same argument as above,

we have $2 \notin D$, that is, $t_2 = s_2$. Then $\{u_1, b_1, a_4, u_4, t_3, a_2, s_1, t_1, u_5, s_4, t_4, b_3, t_2\}$ hosts an F_{12} (with optional edges b_1t_3, a_4t_2). Hence let $4 \notin D$. If $D = \{1\}$, then $\{u_1, a_4, u_4, t_3, s_1, t_1, u_5, t_4, b_3, t_2\}$ hosts an F_{11} (with optional edge a_4t_2). Hence let $D = \{1, 2\}$. Then $u_4 \notin \{a_4, b_4\}$. If $a_4 = b_4$, then $\{u_1, a_4, t_3, s_1, u_5, t_4, t_2\}$ induces an F_9. If $a_4 \neq b_4$, then $\{u_1, a_4, u_4, b_4, t_3, s_1, t_1, a_5, u_5, t_4, b_3, s_2, t_2\}$ hosts an F_{12} (with optional edges a_4s_2, a_4t_2, b_4s_1, b_4t_1). We may now assume that D does not contain 1, and similarly, not 4.
Suppose that $D = \{2\}$. Then we have $u_4 \neq b_4$ and $u_5 \neq a_5$, and $\{u_1, u_4, b_4, t_3, t_1, a_5, u_5, t_4, s_2, t_2\}$ hosts an F_{11} (with optional edges b_4t_1 and t_3a_5). The same holds by symmetry if $D = \{3\}$.
Finally, let $D = \{2, 3\}$. Then we have $u_1 \neq a_1$, $u_4 \neq b_4$, and $u_5 \notin \{a_5, b_5\}$. If $a_5 = b_5$, then $\{u_1, u_4, t_3, t_1, a_5, t_4, s_2\}$ induces an F_9. If $a_5 \neq b_5$, then $\{u_1, u_4, b_4, s_3, t_3, t_1, a_5, u_5, b_5, t_4, s_2, t_2, a_1\}$ hosts an F_{12} (with optional edges a_1t_4, b_4t_1, a_5s_3, a_5t_3, b_5s_2, b_5t_2). This concludes the proof of Case 7.

Case 8: For every i, there is no T_iT_{i+2}-edge.
So each γ_j ($j \in \{1, \dots, 5\}$) satisfies (iv). Thus there is a vertex u_j in R_j that has neighbors t_{j+2} in T_{j+2} and s_{j+3} in T_{j+3}. Suppose that for some j we cannot choose $s_j = t_j$, say for $j = 2$; in other words t_2u_4 and s_2u_5 are not edges and $u_4 \neq b_4$ and $u_5 \neq a_5$. Then $\{u_2, s_2, u_4, u_5, t_2\}$ induces a C_5, so we can apply the algorithm on this C_5, starting from sets $R'_1 = \{u_2\}$, $R'_2 = \{s_2\}$, $R'_3 = \{u_4\}$, $R'_4 = \{u_5\}$, $R'_5 = \{t_2\}$ and growing the corresponding C_5-structure. Note that b_4 and a_5 have neighbors in at least three of $R'_1, \dots, R'_5$. Thus, either one of b_4, a_5 has neighbors in at least four of these sets, and then G contains an F_2 or a J and we can conclude as above; or we have $b_4 \in T'_2$ and $a_5 \in T'_5$ (with the obvious notation), so b_4a_5 is a $T'_2T'_5$-edge and we can argue as in the preceding case. Therefore we may assume that $t_j = s_j$ for each $j \in \{1, \dots, 5\}$. Then $\{u_1, \dots, u_5, t_1, \dots, t_5\}$ induces an F_{11}. This completes the proof of the lemma. □

7.3 When G is 3-colorable

In this section, we analyze the situation when G is 3-colorable. We may assume up to symmetry that the algorithm has found that γ_5 extends to a 3-coloring γ of G. Thus we have $\gamma(R_1 \cup R_4) = 1$, $\gamma(R_2) = 2$, $\gamma(R_3) = 3$, $\gamma(R_5) \subseteq \{2, 3\}$ and we know that $\gamma(W_2 \cup T_1 \cup T_3) = 2$, $\gamma(W_3 \cup T_2 \cup T_4) = 3$, $\gamma(T_5) = 1$, and none of properties (i)–(vi) of Lemma 7.6 hold for $h = 5$, in particular, $W_1 \cup W_4 \cup W_5 = \emptyset$.

For each $i \in \{1, \dots, 5\}$ we define a set $U_i = T_{i-1}^{-T_{i+1}} \cup T_{i+1}^{-T_{i-1}}$. By Lemma 7.3 (11), we know that every vertex of Z is either complete or anticomplete to U_j. Let Z_j be the set of vertices of Z that are anticomplete to U_j.

Lemma 7.8. *Every stable set of G is a subset of one of the nine sets W'_2, W'_3, W''_2, W''_3, T'_2, T'_3, U'_1, U'_4, U'_5, which are defined as follows:*
$W'_2 = W_2 \cup R_2 \cup R_4 \cup T_1 \cup T_3 \cup T_5^{0Z} \cup Z$ *and* $W'_3 = W_3 \cup R_1 \cup R_3 \cup T_2 \cup T_4 \cup T_5^{0Z} \cup Z$,
$W''_2 = W_2 \cup R_2 \cup R_5 \cup T_1 \cup T_3 \cup T_4^{0Z} \cup Z$ *and* $W''_3 = W_3 \cup R_3 \cup R_5 \cup T_2 \cup T_4 \cup T_1^{0Z} \cup Z$,
$T'_2 = R_1 \cup R_4 \cup T_2^{0Z} \cup T_5 \cup Z$ *and* $T'_3 = R_1 \cup R_4 \cup T_3^{0Z} \cup T_5 \cup Z$,
$U'_j = R_j \cup U_j \cup Z_j$, *for* $j \in \{1, 4, 5\}$.

Proof. Let S be any stable set in G. First suppose that S contains no vertex from the W_i's or T_i's. So $S \subseteq R_1 \cup \cdots \cup R_5 \cup Z$. Then $S \subseteq R_j \cup R_{j+2} \cup Z$ for some $j \in \{1, \ldots, 5\}$, and correspondingly S is a subset of one of W'_3, W'_2, W''_3, T'_2, W''_2.

Now suppose that S contains a vertex of W_2. By Lemma 7.3 (5), S contains no vertex from R_1, R_3, T_2, T_4 or T_5. Thus $S \subseteq W_2 \cup R_2 \cup R_4 \cup R_5 \cup T_1 \cup T_3 \cup Z$. If S also contains a vertex from R_4, then S contains no vertex from R_5 and so $S \subseteq W_2 \cup R_2 \cup R_4 \cup T_1 \cup T_3 \cup Z$, which is a subset of W'_2; else $S \subseteq W_2 \cup R_2 \cup R_5 \cup T_1 \cup T_3 \cup Z$, which is a subset of W''_2.

Likewise (by symmetry), when S contains a vertex of W_3, we obtain that S is included in W'_3 or in W''_3.

Therefore we may assume that S contains no vertex of $W_2 \cup W_3$.

Suppose that S contains a vertex of T_1. By Lemma 7.3 (1), S contains no vertex from R_1, T_2, or T_5. Thus $S \subseteq R_2 \cup R_3 \cup R_4 \cup R_5 \cup T_1 \cup T_3 \cup T_4 \cup Z$. If S contains a vertex from R_3, then S contains no vertex from $R_2 \cup R_4$ and, by Lemma 7.3 (1) and (4), S contains no vertex from T_3 or T_1^Z; so $S \subseteq R_3 \cup R_5 \cup T_1^{0Z} \cup T_4 \cup Z \subseteq W''_3$. If S contains a vertex from R_4, then S contains no vertex from $R_3 \cup R_5$ and, by Lemma 7.3 (1) and (4), S contains no vertex from T_4 or T_1^Z; so $S \subseteq R_2 \cup R_4 \cup T_1^{0Z} \cup T_3 \cup Z \subseteq W'_2$. Now let S contain no vertex from $R_3 \cup R_4$. If S contains a vertex from R_2 or T_3, then S contains no vertex from T_4^Z, and so $S \subseteq R_2 \cup R_5 \cup T_1 \cup T_3 \cup T_4^{0Z} \cup Z \subseteq W''_2$. Now let S also contain no vertex from R_2 or T_3. So $S \subseteq R_5 \cup T_1 \cup T_4 \cup Z$. If S contains a vertex of T_1 that is complete to T_4, then S contains no vertex from T_4, so $S \subseteq R_5 \cup T_1 \cup Z \subseteq W''_2$. Likewise, if S contains a vertex of T_4 that is complete to T_1, then S contains no vertex from T_1, so $S \subseteq R_5 \cup T_4 \cup Z \subseteq W''_3$. In the remaining case, we have $S \subseteq R_5 \cup T_1^{-T_4} \cup T_4^{-T_1} \cup Z = R_5 \cup U_5 \cup Z$. If S contains a vertex of $Z \setminus Z_5$ (which is complete to U_5 by Lemma 7.3 (11)), then we have $S \subseteq R_5 \cup Z \subseteq W''_2$. Else, we have $S \subseteq R_5 \cup U_5 \cup Z_5 = U'_5$.

Likewise (by symmetry), when S contains a vertex of T_4, we obtain that S is included in one of W''_2, W'_3, W''_3, U'_5.

Therefore we may assume that S contains no vertex of $T_1 \cup T_4$.

Suppose that S contains a vertex of T_2. By Lemma 7.3 (1), S contains no vertex from R_2 or T_3. Thus $S \subseteq R_1 \cup R_3 \cup R_4 \cup R_5 \cup T_2 \cup T_5 \cup Z$. If S contains a vertex from R_5, then S contains no vertex from R_4, R_1 or T_5, and so $S \subseteq R_3 \cup R_5 \cup T_2 \cup Z \subseteq W_3''$. Now let S contain no vertex from R_5. If S contains a vertex from R_4, then, by Lemma 7.3 (4), S contains no vertex from R_3 or T_2^Z, and so $S \subseteq R_1 \cup R_4 \cup T_2^{0Z} \cup T_5 \cup Z = T_2'$. If S contains a vertex from R_3, then, by Lemma 7.3 (4), S contains no vertex from R_4 or T_5^Z, and so $S \subseteq R_1 \cup R_3 \cup T_5^{0Z} \cup T_2 \cup Z \subseteq W_3'$. Now let S also contain no vertex from $R_3 \cup R_4$. So $S \subseteq R_1 \cup T_2 \cup T_5 \cup Z$. If S contains a vertex of T_2 that is complete to T_5, then S contains no vertex from T_5, so $S \subseteq R_1 \cup T_2 \cup Z \subseteq W_3'$. If S contains a vertex of T_5 that is complete to T_2, then S contains no vertex from T_2, so $S \subseteq R_1 \cup T_5 \cup Z \subseteq T_2'$. In the remaining case, we have $S \subseteq R_1 \cup T_2^{-T_5} \cup T_5^{-T_2} \cup Z = R_1 \cup U_1 \cup Z$. If S contains a vertex of $Z \setminus Z_1$ (which is complete to U_1 by Lemma 7.3 (11)), then we have $S \subseteq R_1 \cup Z \subseteq W_3'$. Else, we have $S \subseteq R_1 \cup U_1 \cup Z_1 = U_1'$.

Likewise (by symmetry), when S contains a vertex of T_3, we obtain that S is included in one of W_2'', T_3', W_2' or U_4'.

Therefore we may assume that S contains no vertex of $T_2 \cup T_3$.

Let S contain a vertex from T_5. Then S contains no vertex from R_5, so $S \subset R_1 \cup R_2 \cup R_3 \cup R_4 \cup T_5 \cup Z$. If S contains a vertex from R_2, then it contains no vertex from R_1, R_3 or T_5^Z, and so $S \subseteq R_2 \cup R_4 \cup T_5^{0Z} \cup Z \subseteq W_2'$. By symmetry, if S contains a vertex from R_3, then $S \subseteq W_3'$. Finally, if S contains no vertex from $R_2 \cup R_3$, then $S \subseteq R_1 \cup R_4 \cup T_5 \cup Z \subseteq T_2'$. This completes the proof of the lemma. □

The preceding lemma shows that computing a maximum stable set in G can be reduced to a fixed number of similar problems on smaller graphs. Moreover, the next lemma shows that each such problem is easy.

Lemma 7.9. *For each set X among the nine sets from Lemma 7.8, the induced subgraph $G[X]$ is bipartite and (consequently) one can compute a maximum weight stable set of $G[X]$ in linear time.*

Proof. First let $X = W_2'$. Then X can be partitioned into the two sets $W_2 \cup R_2 \cup T_1 \cup T_3$ and $R_4 \cup T_5^{0Z} \cup Z$. The first one is a stable set, because all its vertices have color 2. The second one is also a stable set, by the definition of Z and T_5^{0Z}. So $G[X]$ is a bipartite graph, and finding a maximum stable set in $G[X]$ can be done in linear time as explained in Section 2. The same argument holds when X is W_3', by symmetry. Likewise, W_2'' can be partitioned into two stable sets $W_2 \cup R_2 \cup T_1 \cup T_3$ and $R_5 \cup T_4^{0Z} \cup Z$, so $G[W_2'']$ is bipartite. The same holds for W_3'', by symmetry. Similarly, T_2' can be partitioned into the two stable sets $R_1 \cup R_4 \cup T_5$ and $T_2^{0Z} \cup Z$, so $G[T_2']$ is bipartite. The same holds for T_3', by symmetry. Finally, by Lemma 7.3 (11) and the definition of Z_1, the set $U_1' = R_1 \cup T_5^{-T_2} \cup T_2^{-T_5} \cup Z_1$ can be partitioned into the two stable sets $R_1 \cup T_5^{-T_2}$ and $T_2^{-T_5} \cup Z_1$; and the same holds for U_4', by symmetry, and also for U_5'. □

The preceding lemma shows that computing a maximum stable set in G can be reduced to nine instances of the same problem on bipartite subgraphs of G. So the total time complexity of the problem is linear. This completes our proof when G contains a C_5.

As in the preceding section, we know that when the algorithm reaches this point, it has not found any P_5 and it has successfully established that the graph is 3-colorable and solved the maximum weighted stable set problem in linear time. But that does not mean that the graph does not contain any P_5. We clarify this last question in the following lemmas. Let us introduce some new sets. For each $i \in \{1, \ldots, 5\}$, define

$X_i = R_{i-2} \cup T_{i-1} \cup T_{i+1} \cup R_{i+2}$ and
$Y_i = R_i \cup T_i \cup T_{i-2} \cup T_{i+2}$.

Lemma 7.10. *G contains a P_5 if and only if the subgraph induced by one of the fifteen sets W_2', W_2'', W_3', W_3'', X_1, …, X_5, $Z \cup T_5$, Y_1, …, Y_5 contains a $2K_2$.*

Proof. Suppose that one of the fifteen sets X contains a $2K_2$, with vertices a, b, a', b'. First let $X = W_2'$. We know from Lemma 7.9 that X can be partitioned into two stable sets $W_2 \cup R_2 \cup T_1 \cup T_3$ and $R_4 \cup T_5^{0Z} \cup Z$. We know from Lemma 7.3 (8) that some vertex a_1 in R_1 is complete to T_3. Hence a_1 is complete to $W_2 \cup R_2 \cup T_1 \cup T_3$ and anticomplete to $R_4 \cup T_5^{0Z} \cup Z$, and it follows that $\{a_1, a, b, a', b'\}$ induces a P_5 in G. Now let $X = W_2''$. We know that X can be partitioned into two stable sets $W_2 \cup R_2 \cup T_1 \cup T_3$ and $R_5 \cup T_4^{0Z} \cup Z$. By Lemma 7.3 (8), some vertex b_3 in R_3 is complete to T_1, so b_3 is complete to $W_2 \cup R_2 \cup T_1 \cup T_3$ and anticomplete to $R_5 \cup T_4^{0Z} \cup Z$, and $\{b_3, a, b, a', b'\}$ induces a P_5. The same holds when X is equal to W_3' or W_3'', by symmetry. Now let $X = X_1$. We know that X can be partitioned into two stable sets $T_2 \cup R_3$ and $R_4 \cup T_5$. By Lemma 7.3 (8), there is a vertex a_3 in R_3 that is complete to T_5. So $a_3 \notin \{a, b, a', b'\}$, and $\{a_3, a, b, a', b'\}$ induces a P_5. By symmetry, the same holds when X is equal to X_i for any $i \in \{2, \ldots, 5\}$. Now let $X = Z \cup T_5$. Then we can pick any vertex r_5 in R_5 and $\{r_5, a, b, a', b'\}$ induces a P_5. Finally let

$X = Y_1$. By Lemma 7.3 (8), there exists a vertex a_2 in R_2 that is complete to T_4 and there is a vertex b_5 that is complete to T_3. Since R_1 is complete to T_1 and T_3 is complete to T_4, it must be that either the four vertices a, b, a', b' are located one in each of the four sets R_1, T_1, T_3, T_4, or two of a, b, a', b' are in one of R_1, T_1 and the other two are in one of T_3, T_4. In either case, $\{a_2, a, b, a', b'\}$ or $\{b_5, a, b, a', b'\}$ induces a P_5. By symmetry, the same holds when X is equal to Y_i for any $i \in \{2, \ldots, 5\}$.

Conversely, suppose that G contains a P_5 a-b-c-b'-a'. Since the subgraph induced by $R_1 \cup \cdots \cup R_5 \cup Z$ contains no $2K_2$, it must be that at least one of the four vertices a, b, a', b' is in $V(G) \setminus (R_1 \cup \cdots \cup R_5 \cup Z)$, that is, in $W_2 \cup W_3 \cup T_1 \cup \cdots \cup T_5$. Up to symmetry this leads to the following cases.

Case 1: one of a, b is in W_2. Let us call w the vertex in $\{a, b\} \cap W_2$ and y the other vertex in $\{a, b\}$. Since a' and b' are not adjacent to w, they must, by Lemma 7.3 (5), both lie in $R_2 \cup R_4 \cup R_5 \cup W_2 \cup T_1 \cup T_3 \cup Z$. Since a' and b' are adjacent, this leads to the following subcases.
Subcase 1.1: one of a', b' is in Z and the other is in $W_2 \cup T_1 \cup T_3$. Let us call z the vertex in $\{a', b'\} \cap Z$ and t the other vertex in $\{a', b'\}$. Suppose that t is in $W_2 \cup T_1$. Thus t is in $W_2 \cup T_1^Z$. Then, since y is adjacent to w and not to z and t, by Lemma 7.3 (4) or (5) vertex y must be in $R_4 \cup R_5 \cup T_4 \cup Z$. In fact if y is in T_4, then y, t, z violate Lemma 7.3 (5) or (11). If y is in $R_4 \cup Z$, then $\{a, b, a', b'\}$ induces a $2K_2$ in $R_4 \cup Z \cup W_2 \cup T_1$, which is a subset of W_2'. If y is in R_5, then $\{a, b, a', b'\}$ induces a $2K_2$ in $R_5 \cup Z \cup W_2 \cup T_1$, which is a subset of W_2''. Now suppose that t is in T_3, hence in T_3^Z. Since y is adjacent to w and not to z and t, vertex y must be in $R_4 \cup T_5 \cup Z$. In fact, as above, if y is in T_5, then y, t, z violate Lemma 7.3 (11). So y is in $R_4 \cup Z$ and $\{a, b, a', b'\}$ induces a $2K_2$ in $R_4 \cup Z \cup W_2 \cup T_3$, which is a subset of W_2'.
Subcase 1.2: one of a', b' is in R_4 anf the other is in R_5. Since y is adjacent to w and not to a' and b', by Lemma 7.3, y cannot be in any of the sets of the C_5-partition.
Subcase 1.3: one of a', b' is in R_4 and the other is in $W_2 \cup T_1$. Then, by Lemma 7.3 (1) or (5), y must be in $R_4 \cup Z$, so $\{a, b, a', b'\}$ induces a $2K_2$ in $R_4 \cup Z \cup W_2 \cup T_1$, which is a subset of W_2'.
Subcase 1.4: one of a', b' is in R_5 and the other is in $W_2 \cup T_3$. Then, by Lemma 7.3 (1) or (5), y must be in $R_5 \cup Z$, so $\{a, b, a', b'\}$ induces a $2K_2$ in $R_5 \cup Z \cup W_2 \cup T_3$, which is a subset of W_2'. This completes the proof in Case 1.

By symmetry, if one of a, b is in W_3, then $\{a, b, a', b'\}$ induces a $2K_2$ in W_3' or W_3''. We may now assume that none of a, b, a', b' is in $W_2 \cup W_3$.

Case 2: one of a, b is in T_j for some $j \in \{1, \dots, 5\}$. To simplify the notation, put $j = 1$. Let us call u the vertex in $\{a, b\} \cap T_1$ and y the other vertex in $\{a, b\}$. Since a' and b' are not adjacent to u, they must lie in $R_2 \cup R_3 \cup R_4 \cup R_5 \cup T_1 \cup T_3 \cup T_4 \cup Z$. Since a' and b' are adjacent, by Lemma 7.3 (1), we have one of the following subcases.
Subcase 2.1: one of a', b' is in Z. We call this vertex z and call t the other vertex in $\{a', b'\}$. So t is in $T_1 \cup T_3 \cup T_4$. In fact if t is in $T_3 \cup T_4$, then u, t, z violate Lemma 7.3 (11). So t is in T_1, hence in T_1^Z. Since y is adjacent to u and not to t (which are both in T_1), by Lemma 7.3 (1) y is in Z. Then $\{a, b, a', b'\}$ induces a $2K_2$ in $Z \cup T_1$, which is a subset of W_2'. (When $j \in \{1, 2, 3, 4\}$, $Z \cup T_j$ is a subset of W_2' or W_3'; when $j = 5$, recall that $Z \cup T_5$ is one of the fifteen sets of this lemma.)
Subcase 2.2: one of a', b' is in T_1. We call this vertex u' and call y' the other vertex in $\{a', b'\}$. We may assume that $y, y' \notin Z$ (or else we are in Subcase 2.1). Hence y, y' are either both in $R_3 \cup T_4$ or both in $R_4 \cup T_3$, and so $\{a, b, a', b'\}$ induces a $2K_2$ either in $R_3 \cup T_4 \cup T_1$, which is a subset of X_5, or in $R_4 \cup T_3 \cup T_1$, which is a subset of X_2.
Subcase 2.3: one of a', b' is in R_3. We call this vertex u' and call y' the other vertex in $\{a', b'\}$. Then y' is in $R_2 \cup T_3 \cup R_4$. In fact, by Lemma 7.3 (3), y' is not in R_4. Suppose that y' is in R_2. Then, by Lemma 7.3 (1), y must be in T_4, and so $\{a, b, a', b'\}$ induces a $2K_2$ in $T_1 \cup R_2 \cup R_3 \cup T_4$, which is X_5. Now suppose that y' is in T_3. Then y must be in $R_1 \cup T_5$. If y is in R_1, then u is in $T_1^{-R_3}$ and y' is in $T_3^{-R_1}$, so they violate Lemma 7.3 (9). If y is in T_5, then $\{a, b, a', b'\}$ induces a $2K_2$ in $R_3 \cup T_3 \cup T_1 \cup T_5$, which is Y_3. The subcase where one of a', b' is in R_4 is symmetric. The case where $j \in \{2, \dots, 5\}$ is similar. This completes the proof of the lemma. □

The preceding lemma shows that detecting a P_5 in G can be reduced to detecting a $2K_2$ in each of fifteen induced subgraphs of G. Moreover, the next lemma shows that each such problem is easy.

Lemma 7.11. *For each set X among the fifteen sets from Lemma 7.10, we can decide in linear time whether $G[X]$ contains a $2K_2$.*

Proof. When X is any of the sets of Lemma 7.10 different from $Y_1, \dots, Y_5$, we have already observed that $G[X]$ is bipartite, and so detecting a $2K_2$ in $G[X]$ can be solved in linear time as explained in Section 2.

Now assume that $X = Y_i$ for some $i \in \{1, \dots, 5\}$. In this case, the subgraph $G[Y_i]$ is not necessarily bipartite, so we proceed differently. Recall that R_i is complete to T_i and T_{i-2} is complete to T_{i+2}. Let (A, B, C, D) be any permutation of the four sets $R_i, T_i, T_{i-2}, T_{i+2}$; we say that the permutation is *acceptable* if A is complete to B and C is complete to D. For each color j in $\{1, 2, 3\}$, let A_j be the set of vertices of color j in A, and let B_j, C_j, D_j be defined similarly. Recall that in our 3-coloring γ, each of the sets $R_1, \ldots, R_4, T_1, \ldots, T_5$ has only one color, and R_5 can have vertices of color 2 or 3 only. We claim that:

> $G[Y_i]$ contains a $2K_2$ if and only if one of the following holds for some acceptable permutation (A, B, C, D) and some color $j \in \{1, 2, 3\}$:
> (i) Either $G[A_j \cup D]$ or $G[C_j \cup B]$ contains a $2K_2$;
> (ii) B^{-C_j} is not complete to D^{-A_j}.

Let us prove this claim. Clearly, if (i) holds, then $G[Y_i]$ contains a $2K_2$. Now suppose that (ii) holds. Pick non-adjacent vertices b, d with $b \in B^{-C_j}$ and $d \in D^{-A_j}$. So d has a non-neighbor a_j in A_j, and similarly, b has a non-neighbor c_j in C_j. Then $\{a_j, b, c_j, d\}$ induces a $2K_2$ in $G[Y_i]$.
Conversely, suppose that $G[Y_i]$ contains a $2K_2$ with vertices u, v, x, y and edges ux, vy. Since G is 3-colored, we may assume that u, v have the same color j. Suppose that u, v are in the same set A among $R_i, T_i, T_{i-2}, T_{i+2}$. Then x, y are not in A, and not in the set that is complete to A, and since they are adjacent they must be both in one of the remaining two sets. Thus there is an acceptable permutation (A, B, C, D) such that u, v are in A_j and x, y are in D, and (i) holds. Now we may assume that u, v are not in the same set among $R_i, T_i, T_{i-2}, T_{i+2}$, so there is an acceptable permutation (A, B, C, D) such that u is in A and v is in C. Clearly $x \notin A \cup D$ and $y \notin B \cup C$. If $x \in C$, then y must be in A, but then both A and C have two colors, which is impossible as observed above (only R_5 can have two colors). Hence we may assume that $x \in B$ and $y \in D$. Thus (ii) holds for (A, B, C, D). Thus the claim holds.

By the claim, checking whether $G[Y_i]$ contains a $2K_2$ is equivalent to checking whether (i) or (ii) holds for any acceptable permutation and color j. Since each of the subgraphs mentioned in (i) and (ii) is bipartite, we can use the linear time algorithm from Section 2. The argument above implies that there are eight permutations to check and three colors, so the total procedure takes linear time. In fact, given our 3-coloring, it is easy to check that only the following six permutations must be tested: (T_1, R_1, T_3, T_4) with $j = 2$; (T_2, R_2, T_4, T_5) with $j = 3$; (T_3, R_3, T_1, T_5) with $j = 2$; (T_4, R_4, T_2, T_1) with $j = 3$; (R_5, T_5, T_2, T_3) with $j = 3$; (R_5, T_5, T_3, T_2) with $j = 2$. □

8 G contains a $\overline{C}_6$

The proofs and techniques in this section will be quite similar to those in Section 6.2. Let $a_1, a_2, a_3, b_1, b_2, b_3$ be six vertices of G that induce a $\overline{C}_6$, with edges a_1a_2, a_1a_3, a_2a_3, b_1b_2, b_1b_3, b_2b_3, a_1b_1, a_2b_2, a_3b_3. Let us say that a sextuple $(A_1, A_2, A_3, B_1, B_2, B_3)$ is a *$\overline{C}_6$-structure* in G if $A_1, A_2, A_3, B_1, B_2, B_3$ are non-empty and disjoint subsets of $V(G)$ such that (with subscripts modulo 3):

- Each of $A_1, A_2, A_3, B_1, B_2, B_3$ is a stable set,
- The sets A_1, A_2, A_3 are pairwise complete to each other, and the sets B_1, B_2, B_3 are pairwise complete to each other,
- For each $i \in \{1, 2, 3\}$, A_i is complete to B_i and anticomplete to B_{i+2},
- For each $i \in \{1, 2, 3\}$, every vertex of A_i has a non-neighbor in B_{i+1}, and every vertex of B_{i+1} has a non-neighbor in A_i.

Note that the six vertices $a_1, \ldots, b_3$ induce a graph with a $\overline{C}_6$-structure $(\{a_1\}, \{a_2\}, \{a_3\}, \{b_1\}, \{b_2\}, \{b_3\})$. Our aim is to "grow" it to the largest possible structure.

Let H be an induced subgraph of G with a $\overline{C}_6$-structure $(A_1, A_2, A_3, B_1, B_2, B_3)$, and let x be a vertex of $V(G) \setminus V(H)$. We say that x can be *added* to this $\overline{C}_6$-structure if the subgraph induced by $V(H) \cup \{x\}$ has a $\overline{C}_6$-structure obtained by putting x in one of the six sets.

Lemma 8.1. *Let H be an induced subgraph of G with a $\overline{C}_6$-structure $(A_1, A_2, A_3, B_1, B_2, B_3)$. Let x be a vertex of $V(G) \setminus V(H)$ that has a neighbor in $V(H)$. Then either:*
- *G has an induced subgraph that is either a P_5, a C_5, or a member of $\mathcal{F}$,*
- *The subgraph induced by $V(H) \cup \{x\}$ has a $\overline{C}_6$-structure,*
- *There is an integer $h \in \{1, 2, 3\}$ such that x is complete to $A_{h+1} \cup A_{h+2} \cup B_h \cup B_{h+1}$ and anticomplete to $A_h \cup B_{h+2}$.*

Proof. The proof is an algorithm that determines which outcome holds. Using the usual techniques, it is easy to see that this algorithm works in linear time $O(n+m)$. If the algorithm finds an induced subgraph that is a P_5 or a member of $\mathcal{F}$, then it stops with the answer "G is not in class $\mathcal{C}$." If the algorithm finds an induced C_5, then it goes to the preceding step (Section 7). The algorithm works as follows. Let s_A be the number of sets among A_1, A_2, A_3 that contain a neighbor of x, and let s_B be defined similarly. If $s_A = 3$, then x has a neighbor u_i in A_i for each $i \in \{1, 2, 3\}$, and then $\{x, u_1, u_2, u_3\}$ induces a K_4 (a member of $\mathcal{F}$), so the algorithm returns this K_4 and stops. Let us now assume that $s_A \leq 2$ and similarly $s_B \leq 2$.

We claim that $\max\{s_A, s_B\} = 2$. For suppose the contrary, that is, $\max\{s_A, s_B\} \leq 1$. By the hypothesis on x, and up to symmetry, we may assume that x has a neighbor u_1 in A_1, so $s_A = 1$ and x has no neighbor in $A_2 \cup A_3$. Pick any u_2 in A_2 and u_3 in A_3. By the definition of a $\overline{C}_6$-structure, for each $i \in \{1,2,3\}$ vertex u_i has a non-neighbor v_{i+1} in B_{i+1}. Note that $\{u_1, u_2, u_3, v_1, v_2, v_3\}$ induces a $\overline{C}_6$. If x is not adjacent to v_2, then $\{x, u_1, u_2, v_2, v_3\}$ induces either a P_5 or a C_5. So assume that x is adjacent to v_2. If x is not adjacent to v_3, then $\{x, u_1, u_3, v_2, v_3\}$ induces a C_5. So let x be adjacent to v_3; then $s_B = 2$, thus our claim holds.

Since $\max\{s_A, s_B\} = 2$, and up to symmetry, we may assume that $s_A = 2$. So, up to symmetry, x has a neighbor u_i in A_i for each $i \in \{1,2\}$ and x is anticomplete to A_3. Pick any u_3 in A_3. We know that, for each $i \in \{1,2,3\}$, vertex u_i has a non-neighbor v_{i+1} in B_{i+1}. Note that $\{u_1, u_2, u_3, v_1, v_2, v_3\}$ induces a $\overline{C}_6$. If x has a non-neighbor w in B_3, then $\{x, u_1, u_3, w, v_2\}$ induces a P_5 or C_5. So we may assume that x is complete to B_3. Now we distinguish between three cases.

Case 1: x has a neighbor in B_1. Then the argument above concerning the pair (A_1, A_2) can be repeated with the pair (B_3, B_1), so we may assume that x is anticomplete to B_2 and complete to A_2. If x is complete to $A_1 \cup B_1$, then the last outcome of the lemma holds (with $h = 3$). If x is complete to A_1 and not to B_1, then x can be added to A_3. If x is complete to B_1 and not to A_1, then x can be added to B_2. Finally, if x has a non-neighbor u'_1 in A_1 and a non-neighbor v'_1 in B_1, then $\{u'_1, u_2, x, v_3, v'_1\}$ induces a C_5.

Case 2: x has a neighbor in B_2. Then the argument above concerning the pair (A_1, A_2) can be repeated with the pair (B_2, B_3), so we may assume that x is anticomplete to B_1 and complete to A_1. However, the situation now is not symmetric to the situation in Case 1, so we analyze it differently. Let z_2 be a neighbor of x in B_2. We claim that:

$$\text{For each } j \in \{1,2,3\}, \text{ there is no edge between } A_j \text{ and } B_{j+1}. \tag{1}$$

For suppose on the contrary that there is an edge $\alpha_j \beta_{j+1}$ with $\alpha_j \in A_j$ and $\beta_{j+1} \in B_{j+1}$ for some $j \in \{1,2,3\}$. First let $j = 1$. If x is adjacent to β_2, then $\{x, \alpha_1, u_2, \beta_2\}$ induces a K_4 ($= F_1$). If x is not adjacent to β_2, then $\beta_2 \neq v_2$ and by the same argument α_1 is not adjacent to v_2, and then $\{x, v_1, u_2, v_3, \alpha_1, v_2, \beta_2\}$ induces a $\overline{C}_7$ ($= F_9$). Now let $j = 2$. If x is adjacent to α_2, then $\{x, z_2, \alpha_2, \beta_3\}$

induces a K_4. If x is not adjacent to α_2, then $\alpha_2 \neq u_2$ and by the same argument β_3 is not adjacent to u_2, and then $\{x, \alpha_2, u_2, \beta_3, u_1, z_2, u_3\}$ induces a $\overline{C}_7$. Now let $j = 3$. Then $\{x, \alpha_3, z_2, u_1, v_3, u_2, \beta_1\}$ induces a $\overline{C}_7$. Thus claim (1) holds.

By claim (1), $(A_1, A_3, A_2, B_1, B_3, B_2)$ is also a $\overline{C}_6$-structure. Now the situation is entirely similar to that in case 1 and we can conclude as in that case.

Case 3: x has no neighbor in $B_1 \cup B_2$. Thus x is anticomplete to $A_3 \cup B_1 \cup B_2$ and complete to B_3. If x has a non-neighbor u_1' in A_1, then $\{x, u_2, u_1', v_1, v_3\}$ induces a C_5. So let x be complete to A_1. If x has a non-neighbor u_2' in A_2, then we know that u_2' has a non-neighbor v_3' in B_3, and then $\{x, u_1, u_2', v_2, v_3'\}$ induces a C_5. In conclusion, x is complete to $A_1 \cup A_2 \cup B_3$, so x can be added to A_3. □

From an algorithmic point of view, it is easy to test the properties of the preceding lemma. Initially we set $A_i = \{a_i\}$ and $B_i = \{b_i\}$ for each $i \in \{1, 2, 3\}$. Then we apply remarks (a) and (b) from Section 3. So the total running time of this step is $O(m + n)$.

Now we may assume that the algorithm has produced an induced subgraph H of G with a $\overline{C}_6$-structure $(A_1, A_2, A_3, B_1, B_2, B_3)$ with $a_i \in A_i$ and $b_i \in B_i$ for each $i \in \{1, 2, 3\}$ and such that H is maximal with this property in the sense that no vertex can be added to the structure. So the preceding lemma implies that, for every vertex x of $V(G) \setminus V(H)$, either x has no neighbor in $V(H)$ or x satisfies the last item of the lemma. For each $i \in \{1, 2, 3\}$, let X_i be the set of vertices of $V(G) \setminus V(H)$ that are complete to $A_{i+1} \cup A_{i+2} \cup B_i \cup B_{i+1}$ and anticomplete to $A_i \cup B_{i+2}$. Let $Z = V(G) \setminus (V(H) \cup X_1 \cup X_2 \cup X_3)$. Thus $A_1, A_2, A_3, B_1, B_2, B_3, X_1, X_2, X_3, Z$ form a partition of $V(G)$ and there is no edge between Z and $V(H)$. If Z contains two adjacent vertices z, z', then $\{z, z', a_1, b_1\}$ induces a $2K_2$, and the algorithm goes to Section 6. Now let us assume that Z is a stable set.

Lemma 8.2. *Either G contains a P_5 or K_4, or the following properties hold:*

- *Each of X_1, X_2, X_3 is a stable set.*
- *The sets X_1, X_2, X_3 are pairwise complete to each other.*
- *Each vertex of Z is anticomplete to at least one of X_1, X_2, X_3.*
- *$V(G) \setminus Z$ contains no double vertex.*
- *Each double vertex of Z is anticomplete to at least two of X_1, X_2, X_3.*

Proof. Suppose, up to symmetry, that X_1 has two adjacent vertices x_1, x_1'. Then $\{x_1, x_1', a_2, b_2\}$ induces a K_4. The same holds for X_2 and X_3.

Next, suppose, up to symmetry, that there are non-adjacent vertices x_1 in X_1 and x_2 in X_2. Then $\{x_1, a_2, a_1, x_2, b_3\}$ induces a P_5.
Next, suppose that some vertex z in Z has a neighbor x_h in X_h for each $h \in \{1,2,3\}$. By the second item, x_1, x_2, x_3 are pairwise adjacent. Then $\{z, x_1, x_2, x_3\}$ induces a K_4.
Next, suppose that there is a double vertex u in $V(G) \setminus Z$. Up to symmetry, we may assume that $u \in A_1 \cup X_1$. Then $\{u, \overline{u}, a_2, b_2\}$ induces a K_4.
Finally, suppose that some double vertex z in Z has a neighbor x_h in X_h for each $h \in \{1,2\}$. By the second item, $\{z, x_1, x_2, x_3\}$ induces a K_4. □

Again with the usual techniques, the algorithm can test the properties in the preceding lemma in time $O(n+m)$. So let us assume that these properties hold. Then G is 3-colorable, because we can assign color i to the vertices of $A_i \cup B_{i+2} \cup X_i$ for each $i \in \{1,2,3\}$, assign to each simple vertex z in Z the color assigned to any set X_h that contains no neighbor of z, and assign to each double vertex z in Z the two colors assigned to any two sets among X_1, X_2, X_3 that contain no neighbor of z.

Lemma 8.3. *Every stable set of G is a subset of one of the following six sets:*
$A_i \cup B_{i+1} \cup B_{i+2} \cup X_i \cup Z$, $i \in \{1,2,3\}$,
$B_i \cup A_{i+1} \cup A_{i+2} \cup X_{i-2} \cup Z$, $i \in \{1,2,3\}$.
Moreover, each of these six sets induces a bipartite subgraph.

Proof. Let S be any stable set of G. First suppose, up to symmetry, that S contains a vertex from A_1. Then the definition of the $\overline{C}_6$-structure and of the X_i's implies that S contains no vertex from $A_2 \cup A_3 \cup B_1 \cup X_2 \cup X_3$. So S is a subset of $A_1 \cup B_2 \cup B_3 \cup X_1 \cup Z$, which is one of the six sets in the lemma. Moreover this set induces a bipartite graph, because it can be partitioned into the two stable sets $A_1 \cup B_3 \cup X_1$ and $B_2 \cup Z$. Likewise, if S contains a vertex from one of $A_2, \ldots, B_3$, then it is included in one of the other five sets of the lemma, which is bipartite by a similar argument. Finally, suppose that S contains no vertex from $V(H)$. Then the second item in Lemma 8.2 implies that S is a subset of $X_h \cup Z$ for some $h \in \{1,2,3\}$, which is a subset of one of the six sets in this lemma. □

It follows from the preceding lemma that, in order to solve MWSS for G, it suffices to solve it for the six bipartite subgraphs mentioned in the lemma and to return the best solution among the six. By the results of Section 2, this can done in linear time.

If the algorithm reaches this point without finding a P_5, that does not mean that the graph does not contain any P_5. This is clarified by the following lemma.

Lemma 8.4. *G is P_5-free if and only if each of the three subgraphs $G[A_i \cup B_{i+1} \cup X_i \cup Z]$, $i \in \{1, 2, 3\}$ is $2K_2$-free. Moreover, each of these three subgraphs is bipartite.*

Proof. The second sentence of the lemma is obvious, because $A_i \cup X_i$ and $B_{i+1} \cup Z$ are stable sets.

Now we prove the first sentence. Suppose, up to symmetry, that $G[A_1 \cup B_2 \cup X_1 \cup Z]$ contains a $2K_2$, with vertices a, b, c, d and edges ab and cd. Since $A_1 \cup X_1$ and $B_2 \cup Z$ are stable sets, we may assume that $a, c \in A_1 \cup X_1$ and $b, d \in B_2 \cup Z$. Then b-a-a_3-c-d is an induced P_5 in G, for any a_3 in A_3.

Conversely, assume that the three subgraphs contain no induced $2K_2$. For each $i \in \{1, 2, 3\}$, the definition of X_i and Lemma 8.2 imply that every vertex of $V(G) \setminus Z$ is either complete or anticomplete to X_i. Consequently:

(1) *Any vertex that has a neighbor and a non-neighbor in X_i lies in Z.*

Now suppose that G contains an induced P_5 u_1-u_2-u_3-u_4-u_5. Put $X = X_1 \cup X_2 \cup X_3$. First suppose that one of u_1, u_2 is in X, and call this vertex x, say $x \in X_1$. If also one of u_4, u_5 is in X, then, by (1), the other two vertices among u_1, u_2, u_4, u_5 must be in Z, but then these four vertices induce a $2K_2$ in $G[X_1 \cup Z]$, which contradicts the hypothesis. On the other hand, if u_4 and u_5 are not in X, then, since they are not adjacent to x, they must be in $A_1 \cup B_3 \cup Z$, which is impossible as this is a stable set. Thus, and by symmetry, we may assume that none of u_1, u_2, u_4, u_5 is in X. Since the vertices of Z are isolated in $G \setminus X$, vertices u_1, u_2, u_4, u_5 must be in H. Suppose that $u_3 \in X_1$. Then u_1 and u_5 must be in $A_1 \cup B_3$. If $u_1 \in A_1$ and $u_5 \in B_3$ (or vice-versa), then it is easy to see that one cannot place u_2 and u_4 in a way that respects the adjacency on the P_5. So, up to symmetry, let us assume that u_1, u_5 are both in A_1. Then u_2 and u_4 are neither complete nor anticomplete to A_1, so they must be in B_2, but then u_1, u_2, u_4, u_4 induce a $2K_2$ in $G[A_1 \cup B_2]$, which contradicts the hypothesis. So $u_3 \in V(H)$, say $u_3 \in A_3$. Then u_1 and u_5 are in $B_1 \cup B_2$. If they are both in B_1, then it is impossible to place u_2 and u_4 in a way that respects the adjacency on the P_5. Finally, if u_2 and u_4 are both in B_2, then u_2 and u_4 must be in A_1, but then u_1, u_2, u_4, u_5 induce a $2K_2$ in $G[A_1 \cup B_2]$, which contradicts the hypothesis. □

Conclusion

We studied 3-colorable P_5-free graphs and obtained a complete description of their structure. We showed that this structure can always be obtained in linear time. Moreover, using this structure, we showed that the Maximum Weighted Stable Set problem in such a graph G can be reduced to solving it in a fixed number (at most nine) of bipartite subgraphs of G. One may wonder whether a similar situation occurs in the class of k-colorable P_5-free graphs for any fixed k.

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