Les cahiers Leibniz

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Abstract

A b-coloring is a coloring of the vertices of a graph such that each color class contains a vertex that has a neighbor in all other color classes, and the b-chromatic number $b(G)$ of a graph G is the largest integer k such that G admits a b-coloring with k colors. In this paper, we show that every graph $G \neq K_n$ of order n satisfies the inequality $b(G) \leq \left\lfloor \frac{n+\omega(G)-1}{2} \right\rfloor$, where $\omega(G)$ is the size of the maximum clique in G, and we give a characterization of bipartite graphs that achieve equality in the above bound. Also we show that for any graph G, $b(G) - \chi(G) \leq \left\lceil \frac{n}{2} \right\rceil - 2$, and we deduce a characterization of graphs achieving this bound. In [2], the authors showed that if G is a connected graph of order $n \geq 5$, then for any $v \in V(G), b(G-v) \leq b(G) + \left\lfloor \frac{n}{2} \right\rfloor - 2$. We confirm this result for an arbitrary graph of order $n \geq 4$ and we conjecture that this bound is achieved if and only if $G = C_4, P_4$ or $2P_2$.

Keywords: coloring, b-coloring, bipartite graphs

1 Introduction

Let $G = (V, E)$ be a simple graph with vertex-set V and edge-set E. A coloring of the vertices of G is a mapping $c : V \to \{1, 2, \ldots\}$. For every

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vertex $v \in V$ the integer $c(v)$ is called the color of v. A coloring is *proper* if any two adjacent vertices have different colors. The *chromatic number* $\chi(G)$ of graph G is the smallest integer k such that G admits a proper coloring using k colors.

A *b-coloring* of a graph G by k colors is a proper coloring of the vertices of G such that in each color class there exists a vertex having neighbors in all the other $k-1$ colors classes. We call any such vertex a *b-vertex.* The b-chromatic number $b(G)$ of a graph G is the largest integer such that G admits a b-coloring with k colors. The concept of b-coloring has been introduced by R.W. Irving and D.F. Manlove [3, 7]. They proved that determining $b(G)$ is NP-hard for general graphs, even when it is restricted to the class of bipartite graphs [6], but polynomial for trees [3, 7]. The NP-completeness results have incited researchers to establish bounds on the b-chromatic number in general or to find its exact values for subclasses of graphs (see [4, 5, 6]).

For notation and graph theory terminology we in general follow [1]. Consider a graph $G=(V,E)$. For any $A \subset V$, let $G[A]$ denote the subgraph of G induced by A. For any vertex v of G, the *neighborhood* of v is the set $N_G(v)=\{u \in V(G) \mid (u,v) \in E\}$ (or $N(v)$ if there is no confusion), and the *closed neighborhood* of v is the set $N_G[v]=N_G(v)\cup\{v\}$. Let $\omega(G)$ denote the size of a maximum clique of G. If G and H are two vertex-disjoint graphs, the *union* of G and H is the graph $G+H$ whose vertex-set is $V(G)\cup V(H)$ and edge-set is $E(G)\cup E(H)$. For an integer $p \geq 2$, the union of p copies of a graph G is denoted pG. We let P_k denote the path with k vertices, and K_k denote the complete graph with k vertices.

A graph G is called bipartite when its vertex set can be partitioned into two disjoint parts X_1, X_2 such that all edges of G meet both X_1 and X_2. A complete bipartite graph G with bipartition (X_1, X_2) is a bipartite graph such that for any two vertices, $x_1 \in X_1$ and $x_2 \in X_2$, x_1x_2 is an edge in G. The complete bipartite graph with partitions of size $|X_1|=p$ and $|V_2|=q$, is denoted $K_{p,q}$. A set M of independent edges in a graph $G=(V,E)$ is called a matching. M^* is said to be *perfect* if every vertex of G is included in M^*. M_U is a matching of $U \subseteq V$ if every vertex in U is incident with an edge in M.

In this paper, we show that every graph different from clique satisfies the inequality $b(G) \leq \left\lfloor \dfrac{n+\omega(G)-1}{2} \right\rfloor$, and we give a characterization of bipartite graphs that achieve equality in the above bound. Also we show that for any graph G, $b(G)-\chi(G) \leq \left\lceil \dfrac{n}{2} \right\rceil - 2$, and we deduce a characterization of graphs achieving this bound.

2 Upper bound for $b(G)$

In this section we give an upper bound for the b-chromatic number $b(G)$ in terms of n and $\omega(G)$.

Theorem 1 *For every graph $G \neq K_n$ of order $n \geq 2$,*

$$b(G) \leq \left\lfloor \frac{n+\omega(G)-1}{2} \right\rfloor .$$

Proof. It is easy to see that, if $G = K_n$, then $b(K_n) = \dfrac{n+\omega(G)}{2} = n$. Then we may assume that G is not a clique. Let $b(G) = k$ and let c be a b-coloring of G with k colors and let $V_1, V_2, ..., V_k$ be the color classes of c, where V_i consists of the vertices colored i by c for $1 \leq i \leq k$. Let $t \geq 0$ be the number of color classes that contain a single vertex. If $t = 0$, then $n = \displaystyle\sum_{i=1}^{k} |S_i| \geq 2k$, and therefore, $k \leq \dfrac{n}{2} \leq \left\lfloor \dfrac{n+\omega(G)-1}{2} \right\rfloor$. Assume that $t \geq 1$. Without loss of generality, we may suppose that for $i = 1, ..., t$, $|V_i| = 1$. Let x_l any b-vertex of color $l > t$. Then x_l is adjacent to any vertex of $\displaystyle\bigcup_{i=1}^{t} V_i$. It follows that $\left(\displaystyle\bigcup_{i=1}^{t} V_i\right) \cup \{x_l\}$ induces a complete subgraph of G. Thus $\omega(G) \geq t+1$. Since for $i > t$, $|S_i| \geq 2$, it follows that $n = t + \displaystyle\sum_{i=t+1}^{k} |S_i| \geq 2k - t$. This implies that $k \leq \dfrac{n+t}{2} \leq \dfrac{n+\omega(G)-1}{2}$. Since k is a positive integer, then we have the desired inequality. ■

Proposition 2 *For any integer $n \geq 4$, there exists a graph G with*

$$\omega(G) = \lceil \frac{n}{3} \rceil \text{ and } b(G) = \lfloor \frac{n+\omega(G)-1}{2} \rfloor .$$

Proof. In order to construct our graph, let us consider three sets, $A = \{a_1, a_2, ..., a_{\lceil \frac{n}{3} \rceil}\}$, $B = \{b_1, b_2, ..., b_{\lceil \frac{n}{3} \rceil}\}$ $(n \geq 4)$ and $D = \{d_1, d_2, ..., d_r\}$ such that A, B are clique and stable, respectively, with $\left\lceil \dfrac{n}{3} \right\rceil$ vertices, and D is a clique with r vertices, where $r = \begin{cases} \frac{n}{3} \text{ if } n \equiv 0\,[3] \\ \lceil \frac{n}{3} \rceil - 2 \text{ if } n \equiv 1\,[3] \\ \lceil \frac{n}{3} \rceil - 1 \text{ if } n \equiv 2\,[3] \end{cases}$. No vertex of A is adjacent to a vertex of D. We join each vertex $a_i \in A$ to each vertex $b_j \in B$ for each $i \neq j$. Therefore, for $1 \leq i \leq \left\lceil \frac{n}{3} \right\rceil$, $(A \setminus \{a_i\}) \cup \{b_i\}$ forms a clique with $\left\lceil \frac{n}{3} \right\rceil$ vertices. If $n \equiv 0\,\lceil 3 \rceil$, then for every i and every $j \neq r$, we put an edge between any two vertices $b_i \in B$ and $d_j \in D$ and no vertex of B is adjacent to a d_r. If $n \equiv t\,\lceil 3 \rceil$, $t \in \{1, 2\}$, then for every i and every j, we put an edge between any two vertices $b_i \in B$ and $d_j \in D$.

We denote the graph thus constructed by G. It is easy to verify that G satisfies the conditions of this proposition. In fact, according to this construction, we see easily that $\omega(G) = \left\lceil \frac{n}{3} \right\rceil$. In order to show that G satisfies $b(G) = \left\lfloor \frac{n + \omega(G) - 1}{2} \right\rfloor$, it suffices to exhibit a b-coloring G with $\lfloor \frac{n + \omega(G) - 1}{2} \rfloor$-colors. Indeed, For any integer i, color a_i and b_i with i. If $n \equiv 0[3]$, then for every $i \neq r$, color $d_i \in D$ with $\left\lceil \frac{n}{3} \right\rceil + i$, and color d_r with c_0 such that $1 \leq c_0 \leq r$. If $n \equiv t\,\lceil 3 \rceil$, $t \in \{1, 2\}$, then for any integer i, color $d_i \in D$ with $\left\lceil \frac{n}{3} \right\rceil + i$. It is clear that this coloring is proper and b-chromatic. The b-verticers are $B \cup D$ if $t \neq 0$, otherwise $B \cup (D \setminus \{d_r\})$. ■

The following corollary is an immediate consequence of Theorem 1

Corollary 3 *If G is triangle-free graph, then $b(G) \leq \left\lfloor \frac{n+1}{2} \right\rfloor = \left\lceil \frac{n}{2} \right\rceil$.*

Definition 4 *Let G be a graph and c be a b-coloring of G with $b(G)$ colors. A set S of b-vertices of c is said to be b-system of c if $|S| = b(G)$ and for any two vertices x, y of S, $c(x) \neq c(y)$.*

3 Bipartite graphs with $b(G) = \left\lceil \frac{n}{2} \right\rceil$.

We start this section with a straightforward observation.

Observation 5 *Let G be a bipartite graph with bipartition (X_1, X_2). Let c be a b-coloring of G with $b(G)$ colors and let S be a b-system of c.*
For $i = 1, 2$, if $|S \cap X_i| \geq 2$, then all colors of c appear in X_j $(j \neq i)$. Moreover, if $|X_j| = b(G)$, $(j \neq i)$, then all vertices of X_j $(j \neq i)$ have distinct colors.

The following proposition immediately follows from the Corollary 3.

Proposition 6 *Every bipartite graph of order* $n \geq 3$, *satisfies,* $b(G) \leq \left\lceil \frac{n}{2} \right\rceil$.

Next we need two simple lemmas.

Lemma 7 *Let* G *be a bipartite graph with bipartition* (X, Y) *of order* n *such that* $|X| \neq |Y|$. *If there exists a b-coloring of* G *with* $\left\lceil \frac{n}{2} \right\rceil$*-colors, then:*

$$\min(|X|, |Y|) + 1 = \left\lceil \frac{n}{2} \right\rceil.$$

Proof. Suppose there exists a b-coloring of G with $\left\lceil \frac{n}{2} \right\rceil$-colors. Let $b(G) = k = \left\lceil \frac{n}{2} \right\rceil$. Without loss of generality, we may suppose that $|X| > |Y|$. Therefore, $n = |X| + |Y| > 2|Y|$. Thus,

$$|Y| < \frac{n}{2} \leq \left\lceil \frac{n}{2} \right\rceil = k. \tag{1}$$

If $k \geq |Y| + 2$, then X contains at least two b-vertices of distinct colors. Observation 5 implies that all colors appear in Y. Therefore $|Y| \geq \left\lceil \frac{n}{2} \right\rceil$, which contradicts inequality (1). Thus $k = |Y| + 1 = \min(|X|, |Y|) + 1$. ■

Lemma 8 *Let* G *be a bipartite graph with bipartition* (X, Y) *of order* n *such that* $|X| > |Y|$. *If there exists a b-coloring of* G *with* $\left\lceil \frac{n}{2} \right\rceil$*-colors, then all b-vertices of* X *have the same color.*

Proof. Let G be a bipartite graph with bipartition (X, Y) of n such that $|X| > |Y|$ and let $b(G) = k = \left\lceil \frac{n}{2} \right\rceil$. Suppose that X contains at least two b-vertices of distinct colors. Observation 5 implies that all colors appear in Y. Thus $|Y| \geq \left\lceil \frac{n}{2} \right\rceil$. Lemma 7 implies that, $k - 1 = |Y| \geq \left\lceil \frac{n}{2} \right\rceil$, a contradiction to Proposition 6. ■

In order to characterize bipartite graphs having a b-chromatic number equal to $\left\lceil \frac{n}{2} \right\rceil$, we define five bipartite graphs G_i $(i = 1, 2, 3)$ with bipartition (X_i, Y_i), as follows.

1. Graph G_1 is a bipartite graphs with $|X_1| = |Y_1| \geq 3$ such that $X_1 = A \cup C, Y_1 = B \cup D$ where A, B, C and D satisfy the following conditions:

- $|A| = |D| = p$, $|B| = |C| = q \geq p$ and $p + q \geq 3$
- If $p > 1$, then $G[A \cup B] = K_{p,q}$, $G[A \cup D] = K_{p,p} - M^*_{A \cup D}$, $G[B \cup C] = K_{q,q} - M^*_{B \cup C}$ and possibly there is edges from C to D.
- If $p = 1$, $G[A \cup B] = K_{1,q}$, $G[B \cup C] = K_{q,q} - M^*_{B \cup C}$ and possibly there is edges (at most q) from D to X.

If $p = 0$, then $G_1 = K_{q,q} - M^*_{B \cup C}$.

2. Graph G_2 is a bipartite graph with $|X_2| = |Y_2| + 2 \geq 4$ and $|Y_2| \geq 2$ such that $X_2 = \{x\} \cup A \cup \{u, v\}$ and $Y_2 = B \cup \{y\}$ where A, B, x, y, u and v satisfy the following conditions:

- $|A| = |B| = p \geq 1$.
- $N(x) = Y$ and $(A \cup \{x\}) \subseteq N(y)$.
- Each vertex of B has at least one neighbor in $\{u, v\}$.
- $|N(y) \cap \{u, v\}| \leq 1$.
- If y is adjacent to one of u, v, say u, then v is adjacent to all of B.
- If $p = 1$, then there is no edges between A and B, otherwise, $G[A \cup B] = K_{p,p} - M^*_{A \cup B}$.

3. Graph G_3 is a bipartite graph with $|X_3| = |Y_3| + 1 \geq 3$ and $|Y_3| \geq 2$ such that $X_3 = \{x\} \cup A$ where A, x satisfy the following conditions:

- $|A| = p \geq 2$.
- x is adjacent to all vertices of Y.
- $G[A \cup Y] = K_{p,p} - M^*_{A \cup B}$.

Let $H_0 = \{K_1, 2K_1, K_1 + K_2, P_3, 2K_1 + K_2, 2K_2, K_1 + P_3, P_4, K_{1,3}, C_4\}$ and let $\mathcal{F}_b = H_0 \cup \{G_1, G_2, G_3\}$.

Lemma 9 *If $G \in \mathcal{F}_b$, then* $b(G) = \left\lceil \frac{n}{2} \right\rceil$.

Proof. Let $G \in \mathcal{F}_b$. If $G \in H_0$, then it is not hard to verify that, $b(G) = \left\lceil \frac{n}{2} \right\rceil$. Then we can suppose that $G \in \{G_1, G_2, G_3\}$. By Proposition 6, $b(G) \leq \left\lceil \frac{n}{2} \right\rceil$. Thus, to show equality it suffices to give a b-coloring c of G with $\left\lceil \frac{n}{2} \right\rceil$-colors. Now assign a coloring to the vertices of G as follows.
i) Coloring of $G_1 = (X_1, Y_1, E_1)$, where $X_1 = A \cup C, Y_1 = B \cup D$ and $|X_1| = |Y_1|$. Let $B = \{b_1, b_2, ..., b_q\}$ and $C = \{c_1, c_2, ..., c_q\}$. If $|A| \geq 1$, then let $A = \{a_1, a_2, ..., a_p\}$ and $D = \{d_1, d_2, ..., d_p\}$. Color a_i, d_i with i $(1 \leq i \leq p)$ and b_j, c_j with $j + p$ $(1 \leq j \leq q)$. If $|A| = 0$, then color b_j, c_j with j $(1 \leq j \leq q)$.

ii) Coloring of $G_2 = (X_2, Y_2, E_2)$, where $X_2 = \{x\} \cup A \cup \{u, v\}, Y_2 = B \cup \{y\}$ and $|X_2| = |Y_2| + 2$. Let $|A| = |B| = p \geq 1$ and let $A = \{a_1, a_2, ..., a_p\}$, $B = \{b_1, b_2, ..., b_p\}$. Color x with 1 and y with $p + 2$. If y is adjacent to u, then color u with 1 and v with $p + 2$. If y is not adjacent to u, v, then color u, v with $p + 2$. If $p > 1$, then color a_i, b_i with $i + 1$ $(1 \leq i \leq p)$. If $p = 1$, then color a_1, b_1 with 2.

iii) Coloring of $G_3 = (X_3, Y_3, E_3)$, where $X_3 = \{x\} \cup A$ and $|X_3| = |Y_3| + 1$. Let $A = \{a_1, a_2, ..., a_p\}, Y = \{y_1, y_2, ..., y_p\}$ where $p = |A|$. Color x with 1 and a_i, y_i with i $(1 \leq i \leq p + 1)$.

It is easy to verify that this coloring will be a b-coloring with $\left\lceil \frac{n}{2} \right\rceil$-colors. ■

Now we are in a position to prove our main result.

Theorem 10 *Let G be a bipartite graph of order n. Then,*

$$b(G) = \left\lceil \frac{n}{2} \right\rceil \text{ if and only if } G \in \mathcal{F}_b.$$

Proof. Let G be a bipartite graph with bipartition (X, Y) of order n. If $G \in \mathcal{F}_b$, then by Lemma 9, $b(G) = \left\lceil \frac{n}{2} \right\rceil$. To prove the converse, let c be a b-coloring with $\left\lceil \frac{n}{2} \right\rceil$-colors. Let $b(G) = k = \left\lceil \frac{n}{2} \right\rceil$. If $n \leq 4$, then we can verify that the only graphs that satisfy $k = \left\lceil \frac{n}{2} \right\rceil$ are those belonging to H_0. Now we can assume that $n \geq 5$ and therefore $k \geq 3$. Let S be a b-system of c and let $A = S \cap X$ and $B = S \cap Y$. Then $|A| + |B| = k$. There are two cases to consider:

Case 1: n is even. Then $k = \frac{n}{2}, n \geq 6$.

Case 1.1: $|X| = |Y| = k$. Let $X = \{x_1, x_2, ..., x_k\}$ and $Y = \{y_1, y_2, ..., y_k\}$.

Case 1.1.1: $|A| = 0$.

Then all of Y are b-vertices with distinct colors. Observation 5 implies that all vertices of X have distinct colors. Without loss of generality, we may suppose that, $c(x_i) = c(y_i) = i$, for $i = 1$ to k. Each vertex of Y needs $k-1$ distinct colors in its neighborhood. Thus for $i = 1$ to k, y_i is adjacent to all vertives of $X \diagdown \{x_i\}$. Therefore $d_G(x_i) = d_G(y_i) = k-1 (1 \leq i \leq k)$, which gives a complete bipartite graph minus a perfect matching corresponding to a graph G_1.

Case 1.1.2: $|A| = 1$.

Then $|B| = k-1 \geq 2$. By Observation 5 all vertices of X have distinct colors. Without loss of generality, we may suppose $A = \{x_1\}$, $B = \{y_2, y_3, ..., y_k\}$, $c(x_1) = 1$, $c(x_i) = c(y_i) = i$ for $i = 2$ to k and $c(y_1) = l$ where $1 \leq l \leq k$. Each vertex of $B \cup A$ needs $k-1$ distinct colors in its neighborhood. Therefore, for $i = 2$ to k, y_i is adjacent to all vertices of $X \diagdown \{x_i\}$. Hence, x_1 is adjacent to all vertices of B and y_1 is not adjacent to x_l. Vertex y_1 may be adjacent to r $(0 \leq r \leq k-1)$ vertices of X. Thus, $0 \leq d_G(y_1) \leq k-1$, which gives a graph corresponding to G_1.

Case 1.1.3: $|A| = p > 1$ and $|B| = k - p > 1$.

Observation 5 implies that all vertices of X (respectively, Y) have distinct colors. Without loss of generality, we may suppose that $c(x_i) = c(y_i) = i$, for $i = 1$ to k, and $A = \{x_1, x_2, ..., x_p\}$. Therefore $B = \{y_{p+1}, y_{p+2}, ..., y_k\}$. Let $C = X \diagdown A$ and $D = Y \diagdown B$. Then $|A| = |D|$ and $|B| = |C|$. Each b-vertex $x_i \in A$ (respectively, $y_i \in B$) is adjacent to all vertices of $Y \diagdown \{y_i\}$ (respectively, $X \diagdown \{x_i\}$). Hence, the graph obtained is a bipartite graph defined by $X = A \cup C$ and $Y = B \cup D$ where $G[A \cup B] = K_{p,k-p}$, $G[A \cup D]$, $G[B \cup C]$ are two complete bipartite graphs minus a perfect matching, and possibly there is edges from C and D. Thus the resulting graph is isomorphic to G_1.

Case 1.2: $|X| \neq |Y|$.

Without loss of generality, we may suppose that $|X| > |Y|$. Lemma 7 implies that

$$|Y| = k - 1 \geq 2 \tag{2}$$

Since $|X| = n - |Y|$ with $n = 2k$, (2) implies that

$$|X| = |Y| + 2 = k + 1 \geq 4 \tag{3}$$

Let $X = \{x_1, x_2, ..., x_{k+1}\}$ and $Y = \{y_1, y_2, ..., y_{k-1}\}$. Lemma 8 implies that contains only one vertex, say x_1. This implies that $|B| = k-1 \geq 2$. Therefore (2) implies that all vertices of Y are b-vertices with distinct colors. Also x_1 is adjacent to all vertices of Y. Without loss of generality, we may suppose that $c(y_i) = i + 1$ for $i = 1$ to $k - 1$. Therefore $c(x_1) = 1$. Since $|B| \geq 2$, all colors of c appear in X. Hence (3) implies that X contains exactly two vertices of the same color, say x_{k+1}, x_h, where $1 \leq h \leq k$. Without loss of generality, we may suppose that, for $i = 2$ to k, $c(x_i) = i$.

Case 12.1: x_h is a b-vertex

Since $|A| = 1$, $h = 1$ and therefore $c(x_h) = c(x_{k+1}) = 1$. Since all vertices of Y are b-vertices, it follows that, for $i = 1$ to $k - 1$, y_i is adjacent to all of $X \diagdown \{x_{k+1}, x_{i+1}\}$. Vertex x_{k+1} may be adjacent to r $(0 \leq r \leq k - 1)$ vertices of Y. Thus, the resulting graph is isomorphic to G_2.

Case 12.1: x_h is not a b-vertex

Then $h \neq 1$. Suppose that $h = k > 1$. Since $c(y_{k-1}) = k$, y_{k-1} is not adjacent to x_k, x_{k+1}. Since y_{k-1} is a b-vertex, y_{k-1} is adjacent to all vertices of $X \diagdown \{x_k, x_{k+1}\}$. Then, by the same argument, for $i = 1$ to $k - 2$, y_i is adjacent to all of $X \diagdown \{x_k, x_{k+1}, x_{i+1}\}$, also y_i has at least one neighbor in $\{x_h, x_{k+1}\}$. Vertices x_{k+1} and x_k are not adjacent to y_k. Thus the resulting graph is isomorphic to G_2.

Case 2: n is odd.

Then $k = \dfrac{n+1}{2}, n \geq 5$. Without loss of generality, we may suppose that $|X| > |Y|$. Lemma 7 implies that $|Y| = k - 1$. Since $|X| = n - |Y| = n - (k - 1)$ with $n = 2k - 1$, $|X| = |Y| + 1 = k$. Let $X = \{x_1, x_2, ..., x_k\}$ and $Y = \{y_1, y_2, ..., y_{k-1}\}$. Lemma 8 implies that A contains only one vertex, say x_1. Therefore, all of Y are b-vertices with distinct colors ($B = Y$). Vertex x_1 needs $k-1$ distinct colors in its neighborhood. Thus x_1 is adjacent to all vertices of Y. This implies that $c(x_1) \neq c(y_i)$ for $i = 1$ to $k-1$. Without loss of generality, we may suppose that $c(y_i) = i + 1$ for $i = 1$ to $k - 1$. Hence, $c(x_1) = 1$. Since $|B| = k - 1 \geq 2$, all colors of c appear in X. Without loss of generality, we may suppose, for $i = 2$ to k, $c(x_i) = i$. It follows that, for

$i = 1$ to $k-1$, each b-vertex $y_i \in Y$ is adjacent to all vertices of $X \setminus \{x_{i+1}\}$. Hence, the resulting graph is isomorphic to G_3. ■

The following observation is obvious.

Observation 11 *If G is a bipartie graph of order n, then $\alpha(G) \geq \lceil \frac{n}{2} \rceil$.*

The following corollary follows easily from Observation 11.

Corollary 12 *Every bipartite graph of order $n \geq 3$, satisfies $b(G) \leq \alpha(G)$.*

Let $\mathcal{F}_b' = \mathcal{F}_b \setminus \{2K_1, 2K_1 + K_2, K_1 + P_3, K_{1,3}, G_2\}$. Using Observation 11, Corollary 12 and Proposition 6, we can give the following theorem.

Theorem 13 *Let G be a bipartite graphs of order $n \geq 3$. Then $b(G) = \alpha(G)$ if and olny if $G \in \mathcal{F}_b'$.*

Proof. Let $G \in \mathcal{F}_b'$. Then it is easy to see that $b(G) = \alpha(G)$. Suppose that $b(G) = \alpha(G)$. Then $b(G) = \lceil \frac{n}{2} \rceil$. It is not hard to verify that if $G \in \{2K_1, 2K_1 + K_2, K_1 + P_3, K_{1,3}, G_2\}$, then $\alpha(G) \neq \left\lceil \frac{n}{2} \right\rceil$. Thus $G \in \mathcal{F}_b'$. ■

4 $b(G) - \chi(G)$ arbitrarily large

Using Theorem 1, we can deduce the following result:

Theorem 14 *For every graph $G = (V, E)$ of order $n \geq 3$,*

$$b(G) - \chi(G) \leq \left\lceil \frac{n}{2} \right\rceil - 2.$$

With equality if and only if $G = K_3, K_4$ or $G \in \mathcal{F}_b$.

Proof. If $\chi(G) = 1$, then $b(G) - \chi(G) = 0$. Then we may suppose that $\chi(G) \geq 2$. If G is a clique, then this bound is obvious. If G is not a clique, then Theorem 1 implies that $b(G) - \chi(G) \leq \dfrac{n + \omega(G) - 2\chi(G) - 1}{2}$. Since $\omega(G) \leq \chi(G)$, it follows that $b(G) - \chi(G) \leq \dfrac{n - \chi(G) - 1}{2} \leq \dfrac{n-3}{2}$. Thus, $b(G) - \chi(G) \leq \left\lfloor \dfrac{n-3}{2} \right\rfloor = \left\lceil \dfrac{n-4}{2} \right\rceil = \left\lceil \dfrac{n}{2} \right\rceil - 2$.

It is clear that the bounds in Theorem 14 are satisfied with equality for the complete graph K_3 and K_4. Suppose that G is not a clique. If $b(G) - \chi(G) = \left\lceil \frac{n}{2} \right\rceil - 2$, then $\frac{n - \chi(G) - 1}{2} = \frac{n-3}{2}$. Therefore $\chi(G) = 2$. Hence, G is a bipartite graph and $b(G) = \left\lceil \frac{n}{2} \right\rceil$. Thus $G \in \mathcal{F}_b$. ■

Theorem 15 *For each vertex v in a graph G of order $n \geq 4$,*

$$b(G-v) \leq b(G) + \left\lfloor \frac{n}{2} \right\rfloor - 2. \tag{4}$$

Proof. Since $|G - v| \geq 3$, Theorem 14 implies that, $b(G-v) \leq \chi(G-v) + \left\lceil \frac{n-1}{2} \right\rceil - 2$. Since $\chi(G-v) \leq \chi(G) \leq b(G)$ and $\left\lceil \frac{n-1}{2} \right\rceil = \left\lfloor \frac{n}{2} \right\rfloor$, it follows that $b(G-v) \leq b(G) + \left\lfloor \frac{n}{2} \right\rfloor - 2$. ■

Note that S. Francis Raj and R. Balakrishnan [2] have recently found the same upper bound for connected graphs of order at least 5. Here, we have shown that this upper bound remains valid in the general case.

We have seen that if the bound (4) is achieved, then G belongs to the class of graphs satisfying $\chi(G-v) = \chi(G)$, for every vertex v of G.

Observation 16 *Let v be any vertex of a graph G of order $n \geq 4$.*

$$\text{If } b(G-v) = b(G) + \left\lfloor \frac{n}{2} \right\rfloor - 2, \text{ then } \chi(G-v) = \chi(G).$$

Proof. Theorem 14 implies that,

$$b(G-v) \leq \chi(G-v) + \left\lfloor \frac{n}{2} \right\rfloor - 2 \leq \chi(G) + \left\lfloor \frac{n}{2} \right\rfloor - 2 \leq b(G) + \left\lfloor \frac{n}{2} \right\rfloor - 2$$

If $b(G-v) = b(G) + \left\lfloor \frac{n}{2} \right\rfloor - 2$, then $\chi(G-v) = \chi(G)$. ■

Based upon this observation, we state the following conjecture:

Conjecture 17 *For each vertex v in a graph G of order $n \geq 4$,*

$$b(G-v) = b(G) + \left\lfloor \frac{n}{2} \right\rfloor - 2 \text{ if and only if } G = C_4, P_4, 2P_2$$

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