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The hunting of a snark with total chromatic number 5

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The hunting of a snark with total chromatic number 5[1]

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Abstract

A *snark* is a cyclically-4-edge-connected cubic graph with chromatic index 4. In 1880, Tait proved that the Four-Color Conjecture is equivalent to the statement that every planar bridgeless cubic graph has chromatic index 3. The search for counter-examples to the Four-Color Conjecture motivated the definition of the snarks.

A *k-total-coloring* of G is an assignment of k colors to the edges and vertices of G, so that adjacent or incident elements have different colors. The *total chromatic number* χ_T of G is the least k for which G has a k-total-coloring. It is known that the total chromatic number of a cubic graph is either 4 or 5. However, the problem of determining the total chromatic number of a graph is NP-hard even for cubic bipartite graphs.

In 2003, Cavicchioli et al. reported that their extensive computer study of snarks shows that all square-free snarks with less than 30 vertices have total chromatic number 4, and asked for the smallest order of a square-free snark with total chromatic number 5.

In this paper we prove that the total chromatic number of both Blanuša families and an infinite snark family (including the Loupekhine and Goldberg snarks) is 4. Relaxing any of the conditions of cyclic-edge-connectivity and chromatic index, we exhibit cubic graphs with total chromatic number 5.

Keywords: snark, total coloring, edge coloring

1 Introduction

Let $G = (V(G), E(G))$ be a simple connected graph where $V(G)$ is the set of vertices of G and $E(G)$ is the set of edges of G. When there is no chance of ambiguity, we will omit (G) from V and E, and the same convention will be adopted throughout the paper. A graph is said cubic if all its vertices have degree 3.

A *k-edge-coloring* of G is an assignment of k colors to the edges of G so that adjacent edges have different colors. The *chromatic index* χ' of G is

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the least k for which G has a k-edge-coloring. Vizing's theorem [24] states that $\chi' = \Delta$ or $\Delta + 1$, where Δ is the maximum degree of the vertices of G. Graphs with $\chi' = \Delta$ are said to be Class 1, and graphs with $\chi' = \Delta + 1$ are said to be Class 2. The problem of deciding if a graph is Class 1 has been shown NP-complete even for regular graphs of degree at least 3 [15, 18].

A *k-total-coloring* of G is an assignment of k colors to the edges and vertices of G, so that adjacent or incident elements have different colors. The *total chromatic number* χ_T of G is the least k for which G has a k-total-coloring. Clearly, $\chi_T \geq \Delta + 1$ and the Total Coloring Conjecture [24, 1] states that $\chi_T \leq \Delta + 2$. Graphs with $\chi_T = \Delta + 1$ are said to be Type 1, and graphs with $\chi_T = \Delta + 2$ are said to be Type 2.

The problem of deciding if a graph is Type 1 has been shown NP-complete even for cubic bipartite graphs [23, 19]. There are few graph classes whose total chromatic number has been determined. Examples include cycle graphs [26], complete graphs [26], complete bipartite graphs [26], and grids [6]. Another investigation to consider is the validity of the Total Coloring Conjecture. For example, this has been verified for powers of cycles [5] and for cubic graphs [22]. Thus, the total chromatic number of a cubic graph is either 4 or 5.

Coloring is a challenging problem that models many real situations where the adjacencies represent conflicts. In 1880, Tait proved that the Four-Color Conjecture is equivalent to the statement that every planar bridgeless cubic graph has chromatic index 3. The search for counter-examples to the Four-Color Conjecture motivated the definition of the snarks. The importance of these graphs arises partly from the fact that several conjectures would have snarks as minimal counter-examples. This applies to the following three conjectures: Tutte's 5-Flow Conjecture, the 1-Factor Double Cover Conjecture, and the Cycle Double Cover Conjecture [7].

Let G be a cubic graph, let $A \subseteq V$ and let $\omega(A)$ be an edge cutset of size n. If each of $G[A]$ and $G[V - A]$ (the subgraphs of G induced by A and $V - A$) has at least one cycle, $\omega(A)$ is said to be a *c-cutset of size n*. Let G be a graph having at least one c-cutset. The smallest number of edges of a c-cutset of G will be called the *cyclic-edge-connectivity* of G. A graph G is said to be *cyclically-k-edge-connected* if the cyclic-edge-connectivity is at least k.

Snarks are cyclically-4-edge-connected cubic graphs of Class 2. The name snark was given by Gardner [11] based on the poem by Lewis Carroll "The Hunting of the Snark". Isaacs [16] proposed to focus the study of cubic bridgeless graphs of Class 2 on snarks. Indeed, he defined two simple constructions (that we will describe later) such that any Class 2 cubic graph of cyclic-edge connectivity 2 or 3 may be obtained from a smaller Class 2 cubic graph by these constructions. From a Class 2 cubic graph containing a square we can also derive a smaller Class 2 cubic graph but there is no associated construction. Due to this fact, squares are not forbidden in our definition of snarks, unlike in other authors'. An even more restrictive set of Class 2 cubic graphs, the c-minimal snarks, based on other constructions has been proposed by Preissmann [21].

The Petersen graph is the smallest and earliest known snark. It is known that there is no snark of order 12, 14 or 16 (see for example [9, 10]). In [16] Isaacs introduced the dot product, a famous operation used for constructing infinitely many snarks, and defined the Flower snark family. The Blanuša snark of order 18 is constructed using the dot product of two copies of the Petersen graph [2], and Preissmann [20] proved that there are only two snarks of order 18. In this context, Watkins [25] defined two families of snarks constructed using the dot product of Petersen graphs starting from the two snarks of order 18. In addition, the Goldberg and Loupekhine families have been introduced [17, 12].

In [7] Cavicchioli et al. reported that their extensive computer study of snarks shows that all square-free snarks with less than 30 vertices are Type 1, and asked for the smallest order of a Type 2 square-free snark. Later on Brinkmann et al. [3] have shown that this order should be at least 38. In 2011, the infinite families of Flower and Goldberg snarks have had their total chromatic number determined to be 4 [4].

In this paper we prove that an infinite snark family which includes the Loupekhine and Goldberg snarks is Type 1. In addition, we also prove that the Blanuša families, two snark families constructed using the dot product of Petersen graphs, are Type 1.

In the opposite direction, we show that the dot product of Type 1 cubic graphs may be Type 2. Moreover, if we relax any of the conditions of cyclic-edge-connectivity and chromatic index, we can exhibit a Type 2 cubic graph. We give several examples of such graphs.

2 An infinite snark family of Type 1

In this section we define an infinite family of snarks which contains all Goldberg and Loupekhine snarks. We show that all snarks of this family are Type 1. First we give some definitions.

A *zone* is a structure Z consisting of:
- a set V of *vertices*,
- a set E of *edges* with two extremities in V,

- a set S of *semi-edges* with one extremity in V,
- a set N of *null-edges* with no extremity,

such that each vertex is extremity of exactly three elements in $E \cup S$.

An edge of a zone with extremities x and y will be denoted xy, and a semi-edge with extremity x will be denoted by $(x.)$.

Notice that any graph with maximum degree 3 can be made a zone by possibly adding semi-edges and/or null-edges. In particular, any subgraph of a cubic graph corresponds to a unique zone without null-edges. A cubic graph is a zone with no semi- or null-edges.

A *pendant* of a semi- or null-edge e corresponds to one "side" of e which has no extremity. So a semi-edge has one pendant and a null-edge has two. Then the *number of pendants of a zone* is equal to $|S| + 2|N|$. A zone with p pendants will be called a *p-zone*.

By analogy with graphs, a *3-edge-coloring of a zone Z* is an assignment of 3 colors to the edges, semi-edges and null-edges of Z in such a way that every vertex is incident to one element of each color. A pendant has the same color as its corresponding semi- or null-edge.

Similarly a *4-total-coloring of a zone Z* is an assignment of 4 colors to the vertices, edges, semi-edges and null-edges of Z in such a way that in every vertex all four colors are represented.

The following Lemma is well-known:

Lemma 1. *Parity Lemma [2, 8]*

Let c be a 3-edge-coloring of a p-zone by colors $1, 2, 3$ and let p_1, p_2, p_3 be, respectively, the number of pendants colored $1, 2, 3$. Then

$$p_1 \equiv p_2 \equiv p_3 \equiv p \bmod (2).$$

Given two semi-edges $(x.)$ and $(y.)$ of a zone Z, the *junction of* $(x.)$ *and* $(y.)$ consists in replacing $(x.)$ and $(y.)$ by an edge xy. Given one null-edge l and two semi-edges $(x.)$ and $(y.)$ of a zone Z, the *junction of l to* $(x.)$ *and* $(y.)$ consists in replacing l, $(x.)$ and $(y.)$ by an edge xy.

Conversely, given an edge xy of a zone, the *cutting of xy* consists in replacing xy by two semi-edges $(x.)$ and $(y.)$.

The $\mathcal{LG}$-construction

Given $2k+1$ cubic graphs $G_1, G_2, \ldots, G_{2k+1}$ $(k \geq 1)$ and a $(2k+1)$-zone Z we denote by $\mathcal{LG}(G_1, G_2, \ldots, G_{2k+1}, Z)$ the set of cubic graphs obtained as follows (see Figure 1):

- choose in each G_i ($1 \leq i \leq 2k+1$) a chordless path xyz of length 3, cut all five edges connecting x, y, z to the rest of the vertices. We denote by G'_i the so obtained 5-zone containing all vertices of G_i distinct from x, y, z. In G'_i the two semi-edges incident to neighbours (in G_i) of x, respectively z, are called *pairs*, and the semi-edge incident to the neighbour of y is called *lonely*.

- put the G'_i's along a "cycle" by junctions of one pair of G'_i to one pair of G'_{i-1} and of the other pair to one pair of G'_{i+1} (where $G'_0 = G'_{2k+1}$ and $G'_{2k+2} = G'_1$).

- make junctions of the lonely semi-edges of the so obtained $(2k+1)$-zone with the semi- and null-edges of Z.

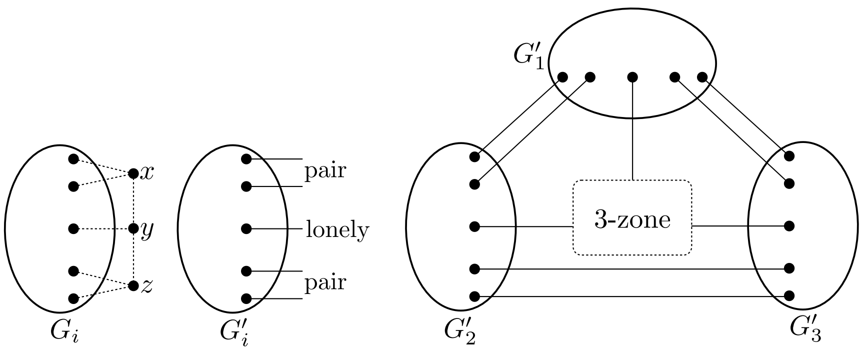


Figure 1: A depiction of the $\mathcal{LG}$-construction (an example with $k = 1$).

The following theorem is well-known in the snarks' folklore.

Theorem 1. *If $G_1, G_2, \ldots, G_{2k+1}$ ($k \geq 1$) are cubic graphs of Class 2 then, for any $(2k+1)$-zone Z, all graphs in $\mathcal{LG}(G_1, G_2, \ldots, G_{2k+1}, Z)$ are Class 2.*

Furthermore, if $G_1, G_2, \ldots, G_{2k+1}$ are snarks, and there is no $W \subseteq V(Z)$ containing a cycle, such that there exist at most three edges or semi-edges having exactly one extremity in W, then all graphs in $\mathcal{LG}(G_1, G_2, \ldots, G_{2k+1}, Z)$ are snarks.

Proof. Let us assume that a graph G in $\mathcal{LG}(G_1, G_2, \ldots, G_{2k+1}, Z)$ is Class 1 and let c be a 3-edge-coloring of G. Then c induces a 3-edge-coloring of each 5-zone G'_i used in the construction of G. By the Parity Lemma this coloring is such that one color appears three times, and the other two only once on the semi-edges of G'_i. It is easy to verify that, since G_i is not 3-edge colorable, one pair of semi-edges of G'_i must be unicolored by c and the other not. Then in the "cycle" of G the colors of the two edges between two consecutive G_i'''s should be alternately equal and distinct. But the "cycle" is odd, so this is not possible, we get a contradiction and conclude that G has no 3-edge-coloring.

To finish the proof of the theorem, we remark that under our conditions the graph G is cyclically-4-edge-connected. □

Two well-known classes of snarks are of the type described in Theorem 1. For both classes, all the G_i's are copies of the Petersen graph P. Due to the high symmetry of P, there is exactly one way to choose a P_3 (path with 3 vertices) in it, so P' is unique. Furthermore, since the two ways for a junction of pairs of two copies of P' are isomorphic, there are exactly two possible "cycles"of $2k + 1$ copies of P' (see Figures 2 and 3). The $(2k + 1)$-zones used to make the graph cubic are specific to each class.

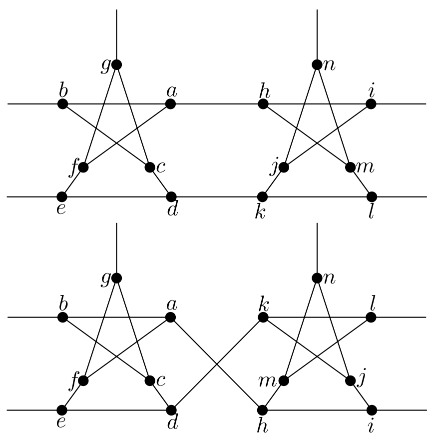


Figure 2: The two ways for a junction of pairs of two copies of P' are isomorphic.

In the case of *Loupekhine snarks*, the $(2k+1)$-zone Z has only one vertex incident to three semi-edges and $k - 1$ null-edges. Any junction of the semi- and null-edges of this zone with the lonely edges of the copies of P' provides a so-called Loupekhine snark (see Figure 3).

In the case of *Goldberg snarks*, the $(2k + 1)$-zone Z is a chordless cycle $v_1, v_2, \dots, v_{2k+1}$ and each v_i is the extremity of a semi-edge $(v_i, .)$. Then the junction of each v_i with the lonely semi-edge of the ith copy of P' provides a so-called Goldberg snark. Notice that for $k = 1$ the condition on Z in Theorem 1 is not respected and so the graphs obtained by the construction are not snarks. In fact, for $k = 1$ the Goldberg snarks are the same as Loupekhine where the vertex of Z is replaced by a triangle.

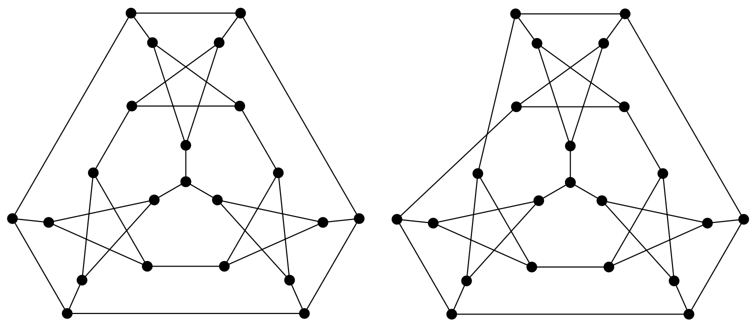


Figure 3: The two snarks of the Loupekhine family for $k = 1$.

Theorem 2. *Let $P_1, P_2, \ldots, P_{2k+1}$ ($k \geq 1$) be copies of the Petersen graph and Z be a $(2k+1)$-zone. If Z has a 4-total-coloring such that at most three colors are used for the extremities of its semi-edges then any graph in $\mathcal{LG}(P_1, P_2, \ldots, P_{2k+1}, Z)$ has a 4-total-coloring.*

Proof. We will show how we can extend a 4-total-coloring of Z to a 4-total-coloring of a graph G in $\mathcal{LG}(P_1, P_2, \ldots, P_{2k+1}, Z)$ using the colorings of P' defined below.

In Figure 4 we have represented five different 4-total-colorings of P'. They all have the property that the semi-edges which are in a pair are colored 1, the extremities of the left pairs are colored by 2 and the extremities of the right pairs are colored 3 and/or 4. So any "cycle" of copies of the Petersen graph using these colorings will be a 4-total-coloring. The colorings of P' represented in Figure 4 differ on the lonely edge and its extremity. We will call $c(i, j)$ a coloring of P' having the properties above and such that the lonely edge is colored by i and its extremity by j. So Figure 4 shows colorings of type $c(1, 2), c(2, 4), c(3, 2), c(2, 3), c(4, 2)$.

Now let us consider a 4-total-coloring c of Z with colors $1, 2, 3, 4$ such that no extremity of a semi-edge receives color 2 (by hypothesis there exists one). We can furthermore choose c such that all null-edges are colored by 2.

We extend c to the whole graph $\mathcal{LG}(P_1, P_2, \ldots, P_{2k+1}, Z)$ as follows.

- for each semi-edge of Z having a junction with the lonely semi-edge of P'_i, if c colors it by $1, 3$ or 4, use respectively $c(1, 2)$, $c(3, 2)$ or $c(4, 2)$ for P'_i, and else use $c(2, 3)$ or $c(2, 4)$ for P_i depending on the color of the extremity of s.

- for each null-edge of Z having a junction to the lonely semi-edges of P'_i and P'_j, use $c(2, 3)$ for P'_i and $c(2, 4)$ for P'_j. □

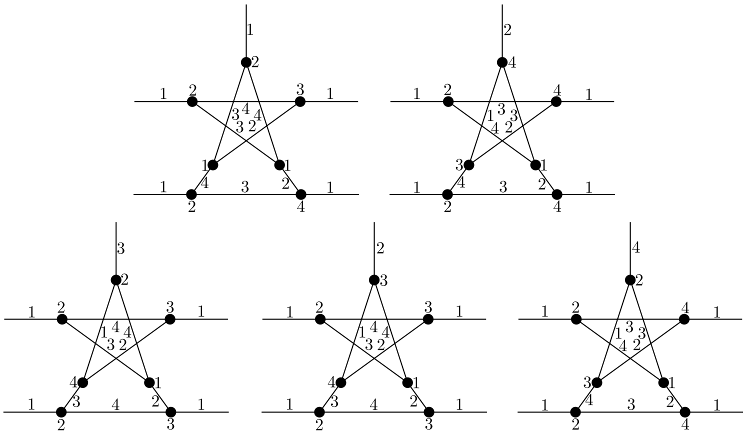


Figure 4: 4-total-colorings of P' of type: $c(1,2), c(2,4), c(3,2), c(2,3), c(4,2)$.

Corollary 1. *All Loupekhine and Goldberg snarks are Type 1.*

Proof. By Theorem 2 it is enough to show that the zones, used by the $\mathcal{LG}$-construction to build the snarks in the Loupekhine and Goldberg families, have a 4-total-coloring such that one of the color does not appear on the vertices that are extremities of a semi-edge.

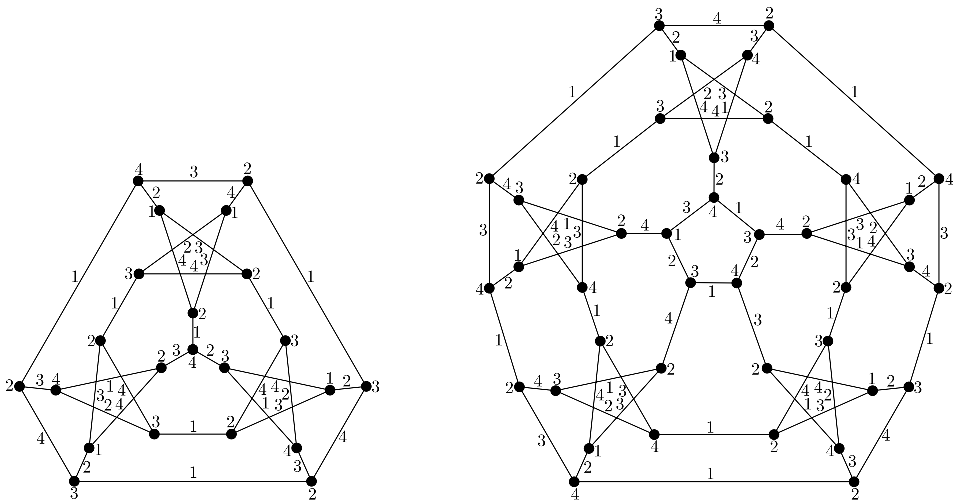
In the case of the Loupekhine family, for any value of $k \geq 1$, this zone has exactly one vertex. Then it is of course 4-totally colorable and only one color is used for the vertex, so we can apply Theorem 2.

In the case of Goldberg family, for each value of $k \geq 2$, the zone Z is a chordless cycle $v_1, v_2, \ldots, v_{2k+1}$ and v_i is the extremity of a semi-edge $(v_i, .)$ for $i = 1 \ldots 2k+1$. We define the following coloring c of the elements of such a zone:

- $c(v_1v_2) = c(v_3v_4) = \ldots = c(v_{2k-1}v_{2k}) = 1$, $c(v_{2k+1}, v_1) = 3$,
- $c(v_2v_3) = c(v_4v_5) = \ldots = c(v_{2k}v_{2k+1}) = 2$,
- $c(v_1) = c(v_3) = \ldots = c(v_{2k-1}) = 4$, $c(v_{2k+1}) = 1$,
- $c(v_2) = c(v_4) = \ldots = c(v_{2k}) = 3$,
- $c((v_2.)) = c((v_4.)) = \ldots = c((v_{2k}.)) = 4 = c((v_{2k+1}.))$,
- $c((v_1.)) = 2$ and $c((v_3.)) = c((v_5.)) = \ldots = c((v_{2k-1}.)) = 3$.

This is a 4-total-coloring of Z not using color 2 for the vertices. Then by Theorem 2, the Goldberg snarks are Type 1. $\square$

Figure 5(a) (resp. (b)) presents a 4-total-coloring of one of the first Loupekhine (resp. Goldberg) snark.



(a) A 4-total-coloring of a smallest Loupekhine snark.

(b) A 4-total-coloring of a smallest Goldberg snark.

Figure 5: The 4-total-colorings obtained by Corollary 1.

3 The Blanuša families are Type 1

Given a cubic graph G and an edge xy of G, the 4-zone not containing x and y that is obtained by cutting the four edges connecting x and y to the other vertices, will be denoted by $G^{-(xy)}$. The two semi-edges corresponding to the edges incident to x, respectively y, are called *a pair of* $G^{-(xy)}$.

Given a cubic graph G and two disjoint edges e and f of G, the 4-zone obtained by cutting e and f will be denoted by G^{ef}. The two semi-edges corresponding to the edge e, respectively f, are called *a pair of* G^{ef}.

A *dot product* of two cubic graphs G and H is any cubic graph obtained from $G^{-(xy)}$ and H^{ef} (for some edges xy of G and e, f of H), by junctions of the pairs of $G^{-(xy)}$ to the pairs of H^{ef} (see Figure 6).

This construction is due to Isaacs [16]. Using the Parity Lemma it is easy to see that the dot product of two Class 2 graphs is Class 2. Furthermore a dot product of two snarks is a snark [16] (but a snark may be a dot product of two Class 2 graphs which are not snarks).

We will show that two infinite families of snarks built by consecutive dot products are Type 1. Before describing these two families defined by Watkins [25] let us consider the dot product of two Petersen graphs.

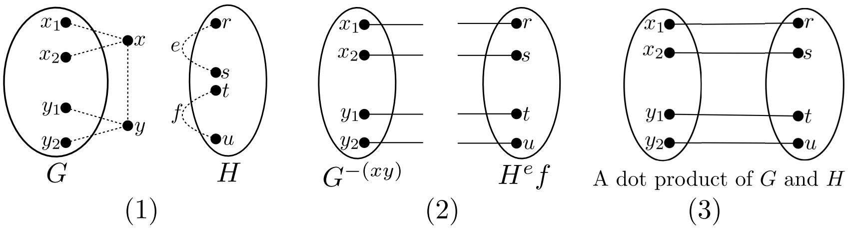


Figure 6: A depiction of the relevant elements of the dot product of G and H.

Due to the high symmetry in the Petersen graph all its edges are equivalent, so there exists only one graph P^{-xy}. Furthermore in P^{-xy} the two pairs are equivalent, and the two semi-edges in a pair are equivalent too (this appears clearly on the drawing of P^{-xy} as in Figure 7). So for any pair of disjoint edges e and f of a cubic graph G, no more than one graph may be obtained by junctions of the pairs of P^{-xy} to the pairs of G^{ef}. Furthermore, in the Petersen graph there are exactly two types of pairs of non adjacent edges (at distance either 1 or 2 from each other). From all preceding remarks we may conclude that there are at most two graphs obtained by a dot product of two Petersen graphs. These are indeed non isomorphic and we will call them Blanuša 1 and Blanuša 2 (Blanuša discovered the first one long before the definition of the dot product). It has been shown that these graphs are the only two snarks of order 18 [20]. They are represented in Figure 8: the top ten vertices correspond to one of the P^{ef} and the eight on the bottom to P^{-xy}. They are the first members respectively of the families $\mathcal{B}_1$ and $\mathcal{B}_2$ proposed by Watkins [25]. It will be convenient for our explanation to use the drawing of these graphs as in Figures 7 and 8. The families are defined recursively as follows: the k-th member of $\mathcal{B}_i$, denoted by $B_i(k)$ ($i \in \{1,2\}$, $k \geq 2$), is obtained by the dot product of P and $B_i(k-1)$ using P^{-xy} and $B_i(k-1)^{ef}$, where e and f are the two edges incident to the bottom vertices of the top copy of P^{ef} in $B_i(k-1)$. We draw it as in Figure 9: the top ten vertices correspond to one of the P^{ef}, the eight vertices on the bottom to P^{-xy}, and between these two subgraphs, there is a vertical chain of $k-1$ copies of P^{-xy}. Each graph in these families is then a snark as it is obtained by the dot product of snarks.

Theorem 3. *All snarks in the families $\mathcal{B}_1$ and $\mathcal{B}_2$ are of Type* 1.

Proof. A 4-total-coloring of $B_i(k)$ ($i \in \{1,2\}, k \geq 2$) is easy to explain using the drawing of Figure 9 and the colorings in Figures 7 and 8. It is obtained by the following way: the subgraph induced by the top ten and the bottom eight vertices is colored as in $B_i(1)$ (see Figure 8), the chain of copies of P^{-xy} is colored using alternately ϕ and ϕ' (see Figure 7), beginning by ϕ for the copy on the top of the chain (see Figure 9). $\square$

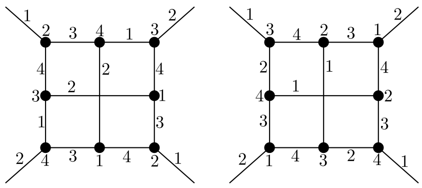


Figure 7: Two 4-total-colorings ϕ and ϕ' of P^{-xy}, respectively.

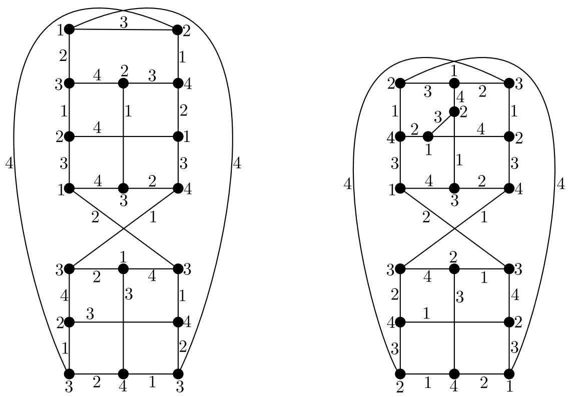


Figure 8: The only two snarks of order 18 and a 4-total-coloring of them.

4 Cubic graphs of Type 2

In the previous sections we presented evidences that all snarks are Type 1. However, if we relax any of the conditions of cyclic-edge-connectivity and chromatic index, we can exhibit a Type 2 cubic graph. In this section we give several examples of such graphs.

To build these examples we use Type 2 graphs of maximum degree 3. The smallest such graphs are K_4 and the complete bipartite $K_{3,3}$ minus an edge that we denote by $K_{3,3} - e$ [14]. For other examples the list made by Hamilton and Hilton [13] of critical such graphs was very useful. The graphs that we will use are represented in Figure 10. We name them H_1, H_2, H_3, H_4.

Remark 1. *A graph G containing a Type 2 subgraph with maximum degree $\Delta(G)$ is Type 2.*

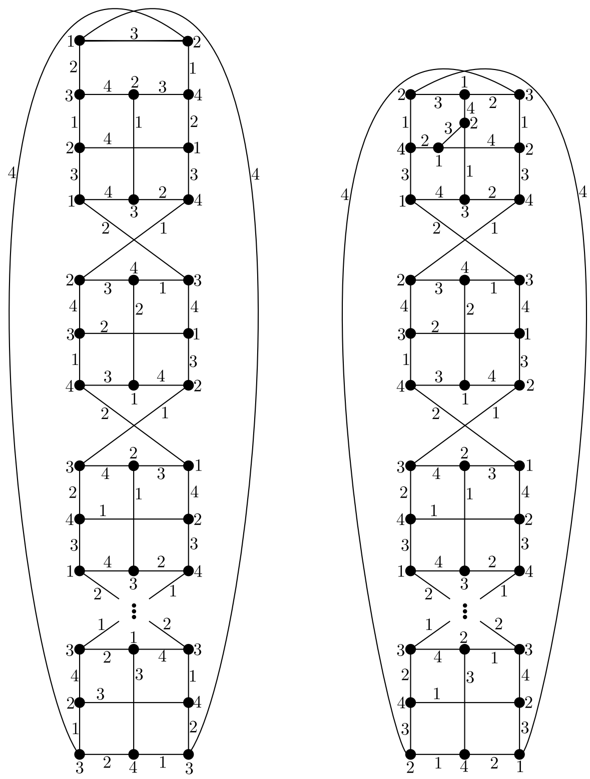


Figure 9: A depiction of the 4-total colorings of both Blanuša families (Theorem 3).

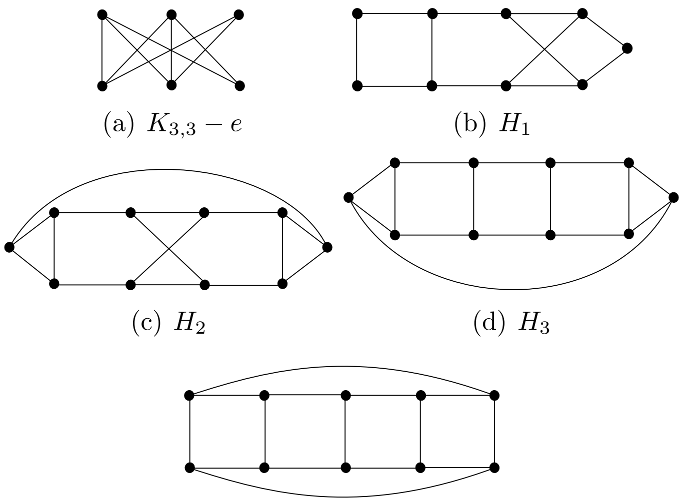
We will use Remark 1, Type 2 graphs of maximum degree 3, and two constructions due to Isaacs [16] to show that for cyclic-edge-connectivity less than 4 there exist cubic graphs of each Class and Type.

4.1 Cubic graphs of cyclic-edge-connectivity 1

It is well-known that all cubic graphs with c-cutset of size 1 are Class 2. Figure 11(a) (resp. (b)) presents an example of a Type 1 (resp. Type 2) cubic graph of cyclic-edge-connectivity 1.

4.2 Cubic graphs of cyclic-edge-connectivity 2

The 2*-construction*: given a cubic graph G and an edge e of G, the 2-zone obtained by cutting e will be denoted by G^e. Given two cubic graphs G and H, any cubic graph obtained from G^e and H^f for some edges e of G and f of H, by junctions of the semi-edges of G^e to the semi-edges of H^f is said to



(a) $K_{3,3} - e$ (b) H_1 (c) H_2 (d) H_3 (e) H_4

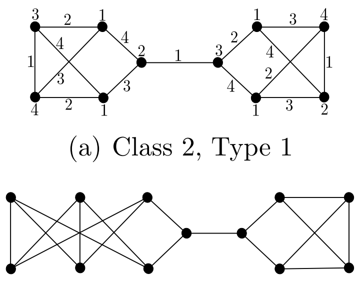
Figure 10: Graphs of maximum degree 3 of Type 2.

be *obtained by a 2-construction* from G and H [16]. We can also specify the edges e and f used for the 2-construction (see Figure 12).

Property 1. *(Isaacs, 1975 [16]) A graph obtained by a 2-construction of cubic graphs G and H is Class 1 if and only if G and H are Class 1.*

Property 2. *Let G be a cubic graph containing an edge e such that G^e is Type 2, and H be any cubic graph. A graph obtained by a 2-construction, based on e, of G and H is Type 2.*

Proof. Direct consequence of Remark 1. □



(a) Class 2, Type 1

(b) Class 2, Type 2

Figure 11: Type 1 (resp. Type 2) cubic graphs of cyclic-edge-connectivity 1.

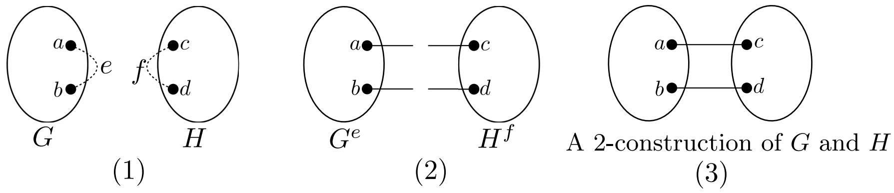


Figure 12: A depiction of the relevant elements of the 2-construction.

We prove that a graph obtained by the 2-construction of Type 1 cubic graphs G and H, is Type 1.

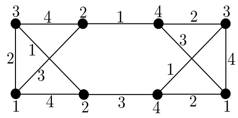
Theorem 4. *Let G and H be cubic graphs of Type 1. Any cubic graph $G2H$ generated from these graphs by a 2-construction is Type 1.*

Proof. Let $e = ab$ and $f = cd$ be the edges of G and H, respectively, that were used for the 2-construction of $G2H$, and let ac and bd be the two edges resulting from this construction.

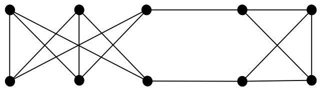
Graphs G and H are Type 1, so each graph has at least one 4-total-coloring with colors $1, 2, 3, 4$. Suppose that π_G and π_H are 4-total-colorings of G and H, respectively. By possibly permutating the colors, we can assume that $\pi_G(e) = \pi_H(f) = 1$, $\pi_G(a) = \pi_H(d) = 2$, and $\pi_G(b) = \pi_H(c) = 3$. Then a 4-total-coloring π of $G2H$ is obtained by keeping the colors given by π_G and π_H and setting $\pi(ac) = \pi(bd) = 1$. $\square$

As a consequence of Theorem 4, and Properties 1 and 2, there exist Class i and Type j cubic graphs of cyclic-edge-connectivity 2, for any $i, j \in \{1, 2\}$.

- Class 1 - Type 1: any 2-construction of two Class 1 and Type 1 cubic graphs, but even the 2-construction of two copies of K_4 (which is Type 2) (see Figure 13 (a)).
- Class 1 - Type 2: any 2-construction of two Class 1 cubic graphs such that at least one of them contains an edge whose deletion leads to a Type 2 graph, for example $K_{3,3}$ and K_4 as in Figure 13 (b).
- Class 2 - Type 1: any 2-construction of two Type 1 cubic graphs such that at least one of them is Class 2, for example P and the *prism* (Figure 17 (a)), as in Figure 14 (a).
- Class 2 - Type 2: any 2-construction of one Class 2 cubic graph with a cubic graph containing an edge whose deletion leads to Type 2 graph, for example P and $K_{3,3}$ as in Figure 14 (b).



(a) Class 1, Type 1



(b) Class 1, Type 2

Figure 13: Cubic graphs of cyclic-edge-connectivity 2, Class 1 and Type 1 (resp. Type 2).

4.3 Cubic graphs of cyclic-edge-connectivity 3

The 3*-construction*: given a cubic graph G and a vertex x of G, the 3-zone not containing x that is obtained by cutting the three edges connecting x to the other vertices, will be denoted by G^{-x}. Given two cubic graphs G and H, any cubic graph obtained from G^{-x} and H^{-y} for some vertices x of G and y of H, by junctions of the semi-edges of G^{-x} to the semi-edges of H^{-y} is said to be *obtained by a* 3*-construction* from G and H [16]. We can also specify the vertices x and y used for the 3-construction (see Figure 15).

Property 3. *(Isaacs, 1975 [16]) A graph obtained by a* 3*-construction of cubic graphs G and H is Class 1 if and only if G and H are Class 1.*

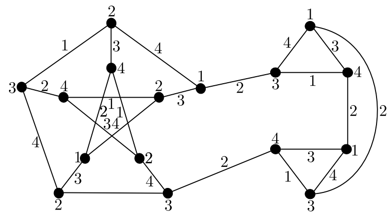
Property 4. *Let G be a cubic graph containing a vertex x such that G^{-x} is Type 2, and H be any cubic graph. A graph obtained by the* 3*-construction, based on x, of G and H is Type 2.*

Proof. Direct consequence of Remark 1. □

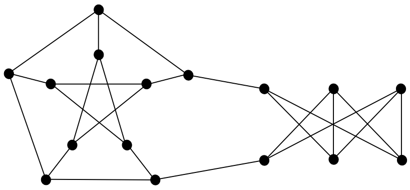
Figure 16 gives an interesting example of a Type 2 cubic graph (H_3, Figure 10 (d)) obtained by a 3-construction of two copies of a prism that is Type 1 (see a 4-total-coloring in Figure 17 (a)). This indicates that there is no analogue to Theorem 4.

We could find Class i and Type j cubic graphs of cyclic-edge-connectivity 2 for any $i, j \in \{1, 2\}$:

- Class 1 - Type 1: the prism (see Figure 17 (a)) is an example of such a graph (notice that the prism is obtained by a 3-construction of two K_4's);



(a) Class 2, Type 1



(b) Class 2, Type 2

Figure 14: Cubic graphs of cyclic-edge-connectivity 2, Class 2 and Type 1 (resp. Type 2).

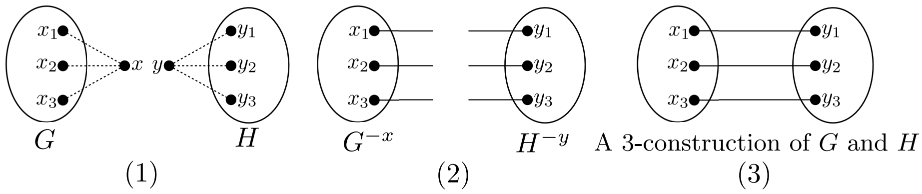
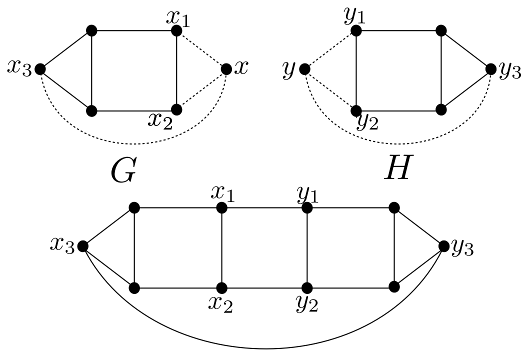


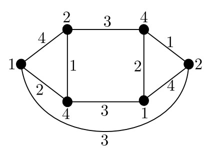
Figure 15: A depiction of the relevant elements of the 3-construction.

- Class 1 - Type 2: the graph H_2 shown in Figure 10 (c) (see also Figure 17 (b));
- Class 2 - Type 1: as shown in Figure 18 (a), the 3-construction of two Petersen graphs is Type 1. From Property 3 we know that it is Class 2;
- Class 2 - Type 2: any 3-construction of one Class 2 cubic graph with a cubic graph containing a vertex whose deletion leads to a Type 2 graph (Property 4). For example P and the cubic graph obtained from the graph H_1 (see Figure 10 (b)) by adding a vertex to each vertex of degree 2 (see this product in Figure 18 (b)). From Property 3 we know that it is Class 2.

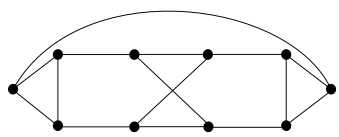


A 3-construction of G and H

Figure 16: The example of a Type 2 cubic graph obtained by the 3-contruction of two copies of a prism.



(a) Class 1, Type 1



(b) Class 1, Type 2

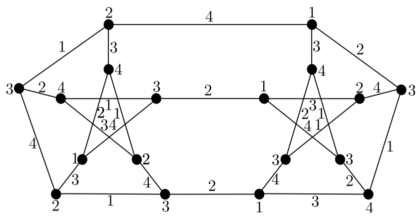
Figure 17: Cubic graphs of cyclic-edge-connectivity 3, Class 1 and Type 1 (resp. Type 2).

4.4 The dot product

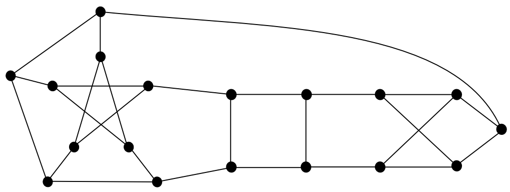
In the previous section we proved that both Blanuša families are Type 1. Recall that these families are constructed using the dot produt of Petersen graphs. However, a dot product of two Type 1 cubic graphs may be Type 2.

Property 5. *A dot product of two cubic graphs of Type 1 may be Type 2.*

Proof. The cubic graph H_4 is an element of the list of Type 2 graphs in Figure 10. Since this graph can be obtained by a dot product of two copies of a prism, we have the result (see Figure 19). □



(a) Class 2, Type 1



(b) Class 2, Type 2

Figure 18: Cubic graphs of cyclic-edge-connectivity 3, Class 2 and Type 1 (resp. Type 2).

5 Conclusion

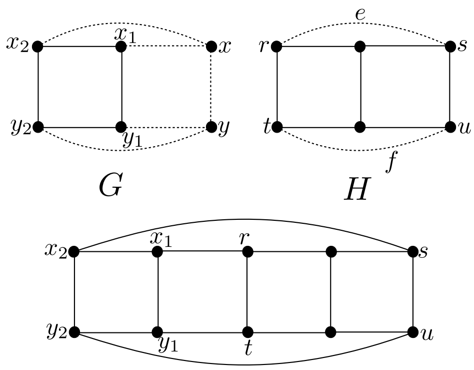
In 2003, Cavicchioli et al. [7] reported that their extensive computer study of snarks shows that all square-free snarks with less than 30 vertices are Type 1, and asked for the smallest order of a Type 2 square-free snark. In 2011, Campos, Dantas and Mello [4] gave two evidences that possibly every snark is Type 1. We contribute additionally to the same direction, since all snarks of this paper are Type 1.

In the opposite direction, if we relax any of the conditions of cyclic-edge-connectivity and chromatic index, we can exhibit several Type 2 cubic graphs. We realized that the total chromatic number seems to have no relation with the chromatic index for a cubic graph of cyclic-edge-connectivity less than 4. Moreover, we observed that all examples of Type 2 cubic graphs have a square. Therefore, we propose the main question.

Question 1. *What is the smallest Type 2 cubic graph without a square?*

The dot product of the two prisms shown in Figure 19 is a Type 2 cubic graph of cyclic-edge connectivity 4, and is Class 1. However, we have not succeeded in finding any Type 2 cubic graph of cyclic-edge connectivity 4 which is Class 2, even with a square. Therefore, we can extend Cavicchioli's problem to all snarks.

Question 2. *What is the smallest Type 2 snark?*



A dot product of G and H

Figure 19: An example of a Type 2 graph obtained by the dot product of two copies of a prism.

Acknowledgements

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References

[1] M. Behzad, G. Chartrand, and J. K. Cooper Jr. The colour numbers of complete graphs. *J. Lond. Math. Soc.*, 42:226–228, 1967.

[2] D. Blanuša. Problem cetiriju boja (in russian). *Glasnik Mat. Fiz. Astr. Ser. II*, 1:31–42, 1946.

[3] G. Brinkmann, J. Goedgebeur, J. Hägglund, and K. Markström. Generation and properties of snarks. In preparation, 2011.

[4] C. N. Campos, S. Dantas, and C. P. Mello. The total-chromatic number of some families of snarks. *Discrete Math.*, 311:984–988, 2011.

[5] C. N. Campos and C. P. Mello. A result on the total colouring of powers of cycles. *Discrete Appl. Math.*, 155:585–597, 2007.

[6] C. N. Campos and C. P. Mello. The total chromatic number of some bipartite graphs. *Ars Combin*, 88:335–347, 2008.

[7] A. Cavicchioli, T. E. Murgolo, B. Ruini, and F. Spaggiari. Special classes of snarks. *Acta Appl. Math.*, 76:57–88, 2003.

[8] B. Descartes. Network-colourings. *Math. Gazette*, 32:67–69, 1948.

[9] S. Fiorini and R. J. Wilson. Edge-colourings of graphs. *Res. Notes Math.*, 16:49, 1977.

[10] J. L. Fouquet. Note sur la non existence d' un snark d' ordre 16. *Discrete Math.*, 38:163–171, 1982.

[11] M. Gardner. Mathematical games: Snarks, Boojums and other conjectures related to the four-color-map theorem. *Sci. Am.*, 234 No. 4:126–130, 1976.

[12] M. K. Goldberg. Construction of class 2 graphs with maximum vertex degree 3. *J. Comb. Theory Ser. B*, 31(3):282–291, 1981.

[13] G. M. Hamilton and A. J. W. Hilton. Graphs of maximum degree 3 and order at most 16 which are critical with respect to the total chromatic number. *J. Combin. Math. Combin. Comput.*, 10:129–149, 1991.

[14] A. J. W. Hilton. The total chromatic number of nearly complete bipartite graphs. *J. Combin. Theory Ser. B*, 52:9–19, 1991.

[15] Ian Holyer. The NP-completeness of edge-colouring. *SIAM J. Comput.*, 10:718–720, 1981.

[16] Rufus Isaacs. Infinite families of nontrivial trivalent graphs which are not Tait colorable. *Amer. Math. Monthly*, 82(3):221–239, March 1975.

[17] Rufus Isaacs. Loupekhine's snarks: a bifamily of non-Tait-colorable graphs. Technical Report 263, Dpt. of Math. Sci., The Johns Hopkins University, Maryland, U.S.A., 1976.

[18] D. Leven and Z. Galil. NP-completeness of finding the chromatic index of regular graphs. *J. Algorithms*, 4:35–44, 1983.

[19] C.J.H. McDiarmid and A. Sánchez-Arroyo. Total colouring regular bipartite graphs is NP-hard. *Discrete Math.*, 124:155–162, 1994.

[20] M Preissmann. Snarks of order 18. *Discrete Math.*, 42:125–126, 1982.

[21] M Preissmann. C-minimal snarks. *Ann. Discrete Math.*, 17:559–565, 1983.

[22] M. Rosenfeld. On the total chromatic number of a graph. *Israel J. Math.*, 9:396–402, 1971.

[23] A. Sánchez-Arroyo. Determining the total colouring number is NP-hard. *Discrete Math.*, 78:315–319, 1989.

[24] V. G. Vizing. On an estimate of the chromatic class of a p-graph. *Metody Diskret. Analiz.*, 3:25–30, 1964.

[25] J. J. Watkins. On the construction of snarks. *Ars Combin.*, 16-B:111–123, 1983.

[26] H. P. Yap. *Total colourings of graphs.* in: Lecture Notes in Mathematics, Springer-Verlag, Germany, 1996.

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