Packing of Rigid Spanning Subgraphs and Spanning Trees

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Abstract

We prove that every \((6k + 2\ell, 2k)\)-connected simple graph contains \(k\) rigid and \(\ell\) connected edge-disjoint spanning subgraphs. This implies a theorem of Jackson and Jordán [4] and a theorem of Jordán [6] on packing of rigid spanning subgraphs. Both these results are generalizations of the classical result of Lovász and Yemini [9] saying that every 6-connected graph is rigid for which our approach provides a transparent proof. Our result also gives two improved upper bounds on the connectivity of graphs that have interesting properties: (1) every 8-connected graph packs a spanning tree and a 2-connected spanning subgraph; (2) every 14-connected graph has a 2-connected orientation.

1 Definitions

Let \(G = (V, E)\) be a graph. We will use the following connectivity concepts. \(G\) is called connected if for every pair \(u, v\) of vertices there is a path from \(u\) to \(v\) in \(G\). \(G\) is called \(k\)-edge-connected if \(G - F\) is connected for all \(F \subseteq E\) with \(|F| \leq k - 1\). \(G\) is called \(k\)-connected if \(|V| > k\) and \(G - X\) is connected for all \(X \subseteq V\) with \(|X| \leq k - 1\). For a pair of positive integers \((p, q)\), \(G\) is called \((p, q)\)-connected if \(G - X\) is \((p - q|X|)\)-edge-connected for all \(X \subset V\). By Menger theorem, \(G\) is \((p, q)\)-connected if and only if for every pair of disjoint subsets \(X, Y\) of \(V\) such that \(Y \neq \emptyset, X \cup Y \neq V\),

\[
d_{G - X}(Y) \geq p - q|X|. \tag{1}
\]

For a better understanding we mention that \(G\) is \((6, 2)\)-connected if \(G\) is 6-edge-connected, \(G - v\) is 4-edge-connected for all \(v \in V\) and \(G - u - v\) is 2-edge-connected for all \(u, v \in V\). It follows from the definitions that \(k\)-edge-connectivity is equivalent to \((k, k)\)-connectivity. Moreover, since loops and parallel edges do not play any role in vertex connectivity, every \(k\)-connected graph contains a \((k, 1)\)-connected simple spanning subgraph. Note also that \((k, 1)\)-connectivity implies \((k, q)\)-connectivity for all \(q \geq 1\). (Remark that this connectivity concept is (very slightly) different from the one introduced by Kaneko and Ota [7] since \(p\) is not required to be a multiple of \(q\).)
Let $D = (V, A)$ be a directed graph. $D$ is called strongly connected if for every ordered pair $(u, v) \in V \times V$ of vertices there is a directed path from $u$ to $v$ in $D$. $D$ is called $k$-arc-connected if $G - F$ is strongly connected for all $F \subseteq A$ with $|F| \leq k - 1$. $D$ is called $k$-connected if $|V| > k$ and $G - X$ is strongly connected for all $X \subset V$ with $|X| \leq k - 1$.

For a set $X$ of vertices and a set $F$ of edges, denote $G_F$ the subgraph of $G$ on vertex set $V$ and edge set $F$, that is $G_F = (V, F)$ and $E(X)$ the set of edges of $G$ induced by $X$. Denote $\mathcal{R}(G)$ the rigidity matroid of $G$ on ground-set $E$ with rank function $r_{\mathcal{R}}$ (for a definition we refer the reader to [9]). For $F \subseteq E$, by a theorem of Lovász and Yemini [9],

$$r_{\mathcal{R}}(F) = \min \sum_{X \in \mathcal{H}} (2|X| - 3),$$

where the minimum is taken over all collections $\mathcal{H}$ of subsets of $V$ such that $(E(X) \cap F, X \in \mathcal{H})$ partitions $F$.

**Remark 1.** If $\mathcal{H}$ achieves the minimum in (2), then each $X \in \mathcal{H}$ induces a connected subgraph of $G_F$.

We will say that $G$ is rigid if $r_{\mathcal{R}}(E) = 2|V| - 3$.

## 2 Results

Lovász and Yemini [9] proved the following sufficient condition for a graph to be rigid.

**Theorem 1** (Lovász and Yemini [9]). Every 6-connected graph is rigid.


**Theorem 2** (Jackson and Jordán [4]). Every (6, 2)-connected simple graph is rigid.


**Theorem 3** (Jordán [6]). Let $k \geq 1$ be an integer. Every $6k$-connected graph contains $k$ edge-disjoint rigid spanning subgraphs.

The main result of this paper contains a common generalization of Theorems 2 and 3. It provides a sufficient condition to have a packing of rigid spanning subgraphs and spanning trees.

**Theorem 4.** Let $k \geq 1$ and $\ell \geq 0$ be integers. Every $(6k + 2\ell, 2k)$-connected simple graph contains $k$ rigid spanning subgraphs and $\ell$ spanning trees pairwise edge-disjoint.

Note that in Theorem 2, the connectivity condition is the best possible since there exist non-rigid (5, 2)-connected graphs (see [9]) and non-rigid (6, 3)-connected graphs, for an example see Figure 1.
Let us see some corollaries of the previous results. Theorem 4 applied for \(k = 1\) and \(\ell = 0\) provides Theorem 2. Since \(6k\)-connectivity implies \((6k, 2k)\)-connectivity of a simple spanning subgraph, Theorem 4 implies Theorem 3.

One can easily derive from the rank function of \(R(G)\) that rigid graphs with at least 3 vertices are 2-connected (see Lemma 2.6 in [5]). Thus, Theorem 4 gives the following corollary.

**Corollary 1.** Let \(k \geq 1\) and \(\ell \geq 0\) be integers. Every \((6k + 2\ell, 2k)\)-connected simple graph contains \(k\) 2-connected and \(\ell\) connected edge-disjoint spanning subgraphs.

Corollary 1 allows us to improve two results of Jordán. The first one deals with the following conjecture of Kriesell, see in [6].

**Conjecture 1 (Kriesell).** For every positive integer \(\lambda\) there exists a (smallest) \(f(\lambda)\) such that every \(f(\lambda)\)-connected graph \(G\) contains a spanning tree \(T\) for which \(G - E(T)\) is \(\lambda\)-connected.

As Jordán pointed out in [6], Theorem 3 answers this conjecture for \(\lambda = 2\) by showing that \(f(2) \leq 12\). Corollary 1 applied for \(k = 1\) and \(\ell = 1\) directly implies that \(f(2) \leq 8\).

**Corollary 2.** Every 8-connected graph \(G\) contains a spanning tree \(T\) such that \(G - E(T)\) is 2-connected.

The other improvement deals with the following conjecture of Thomassen [10].

**Conjecture 2 (Thomassen [10]).** For every positive integer \(\lambda\) there exists a (smallest) \(g(\lambda)\) such that every \(g(\lambda)\)-connected graph \(G\) has a \(\lambda\)-connected orientation.

By applying Theorem 3 and an orientation result of Berg and Jordán [1], Jordán proved in [6] the conjecture for \(\lambda = 2\) by showing that \(g(2) \leq 18\).
Corollary 1 allows us to prove a general result that implies $g(2) \leq 14$. For this purpose, we use a result of Király and Szigeti [8].

**Theorem 5** (Király and Szigeti [8]). An Eulerian graph $G = (V, E)$ has an Eulerian orientation $D$ such that $D - v$ is $k$-arc-connected for all $v \in V$ if and only if $G - v$ is $2k$-edge-connected for all $v \in V$.

Corollary 1 and Theorem 5 imply the following corollary which gives the claimed bound for $k = 1$.

**Corollary 3.** Every simple $(12k + 2, 2k)$-connected graph $G$ has an orientation $D$ such that $D - v$ is $k$-arc-connected for all $v \in V$.

**Proof.** Let $G = (V, E)$ be a simple $(12k + 2, 2k)$-connected graph. By Theorem 5 it suffices to prove that $G$ contains an Eulerian spanning subgraph $H$ such that $H - v$ is $2k$-edge-connected for all $v \in V$. By Corollary 1, $G$ contains $2k$ 2-connected spanning subgraphs $H_i = (V, E_i), i = 1, \ldots, 2k$ and a spanning tree $F$ pairwise edge-disjoint. Define $H' = (V, \bigcup_{i=1}^{2k} E_i)$. For all $i = 1, \ldots, 2k$, since $H_i$ is 2-connected, $H_i - v$ is connected; hence $H' - v$ is $2k$-edge-connected for all $v \in V$. Denote $T$ the set of vertices of odd degree in $H'$. We say that $F'$ is a $T$-join if the set of odd degree vertices of $G_{F'}$ coincides with $T$. It is well-known that the connected graph $F$ contains a $T$-join. Thus adding the edges of this $T$-join to $H'$ provides the required spanning subgraph of $G$.

Finally we mention that the following conjecture of Frank, that would give a necessary and sufficient condition for a graph to have a 2-connected orientation, would imply that $g(2) \leq 4$.

**Conjecture 3** (Frank [3]). A graph has a 2-connected orientation if and only if it is $(4, 2)$-connected.

### 3 Proofs

To prove Theorem 4 we need to introduce two other matroids on the edge set $E$ of $G$. Denote $\mathcal{C}(G)$ the **circuit matroid** of $G$ on ground-set $E$ with rank function $r_C$ given by (3). Let $n$ be the number of vertices in $G$, that is $n = |V|$. For $F \subseteq E$, denote $c(G_F)$ the number of connected components of $G_F$, it is well known that,

$$r_C(F) = n - c(G_F). \tag{3}$$

To have $k$ rigid spanning subgraphs and $\ell$ spanning trees pairwise edge-disjoint in $G$, we must find $k$ basis in $R(G)$ and $\ell$ basis in $C(G)$ pairwise disjoint. To do that we will need the following matroid. For $k \geq 1$ and $\ell \geq 0$, define $\mathcal{M}_{k, \ell}(G)$ as the matroid on ground-set $E$, obtained by taking the matroid union of $k$ copies of the rigidity matroid $R(G)$ and $\ell$ copies of the circuit matroid $C(G)$. Let $r_{\mathcal{M}_{k, \ell}}$ be the rank function of $\mathcal{M}_{k, \ell}(G)$. By a theorem of Edmonds [2], for the rank of matroid unions,

$$r_{\mathcal{M}_{k, \ell}}(E) = \min_{F \subseteq E} kr_R(F) + \ell r_C(F) + |E \setminus F|. \tag{4}$$

In [6], Jordán used the matroid $\mathcal{M}_{k, 0}(G)$ to prove Theorem 3 and pointed out that using $\mathcal{M}_{k, \ell}(G)$ one could prove a theorem on packing of rigid spanning
subgraphs and spanning trees. We tried to fulfill this gap by following the proof of [6] but we failed. To achieve this aim we had to find a new proof technique. Let us first demonstrate this technique by giving a transparent proof for Theorems 1 and 2.

**Proof of Theorem 1.** By (2), there exists a collection \( \mathcal{G} \) of subsets of \( V \) such that \( \{E(X), X \in \mathcal{G}\} \) partitions \( E \) and \( r_{\mathcal{R}}(E) = \sum_{X \in \mathcal{G}} (|X| - 3) \). If \( V \notin \mathcal{G} \) then \( r_{\mathcal{R}}(E) \geq 2|V| - 3 \) hence \( G \) is rigid. So in the following we may assume that \( V \notin \mathcal{G} \).

Let \( \mathcal{H} = \{X \in \mathcal{G} : |X| \geq 3\} \) and \( F = \bigcup_{X \in \mathcal{H}} E(X) \). We define, for \( X \in \mathcal{H} \), the border of \( X \) as \( X_B = X \cap (\bigcup_{Y \in \mathcal{H} \setminus X} Y) \) and the proper part of \( X \) as \( X_I = X \setminus X_B \) and \( \mathcal{H}' = \{X \in \mathcal{H} : X_I \neq \emptyset\} \).

Since every edge of \( F \) is induced by an element of \( \mathcal{H} \), for \( X \in \mathcal{H}' \), by definition of \( X_I \), no edge of \( F \) contributes to \( d_{G - X_B}(X_I) \); and for a vertex \( v \in V - V(\mathcal{H}) \), no edge of \( F \) contributes to \( d_G(v) \). Thus, since for \( X \in \mathcal{H}' \), \( X_I \neq \emptyset \) and \( X_I \cup X_B = X \neq V \), by 6-connectivity of \( G \), we have \( |E \setminus F| \geq \frac{1}{2}(\sum_{X \in \mathcal{H}} d_{G - X_B}(X_I) + \sum_{v \in V - V(\mathcal{H})} d_G(v)) \geq \frac{1}{2}(\sum_{X \in \mathcal{H}} (6 - |X_B|) + \sum_{v \in V - V(\mathcal{H})} |X_B| + 3(V - |V(\mathcal{H})|)) \).

Since for \( X \in \mathcal{H} \setminus \mathcal{H}' \), \( |X_B| = |X| \geq 3 \), we have \( \sum_{X \in \mathcal{H}} (2|X| - 3) = \sum_{X \in \mathcal{H}} 2|X| - 3|\mathcal{H}| + 3|\mathcal{H}'| - 3|\mathcal{H}'| \geq \sum_{X \in \mathcal{H}} 2|X| - 3|\mathcal{H}||X_B| - 3|\mathcal{H}'| \).

Since \( G \) is simple, by Remark 1 every \( X \in \mathcal{G} \) of size 2 induces exactly one edge. Hence, by the above inequalities, we have \( \sum_{X \in \mathcal{G}} (2|X| - 3) = \sum_{X \in \mathcal{H}} (2|X| - 3) + |E \setminus F| \geq \sum_{X \in \mathcal{H}} 2|X| - \sum_{X \in \mathcal{H}} |X_B| + 3(|V| - |V(\mathcal{H})|) = (\sum_{X \in \mathcal{H}} 2|X_I| + \sum_{X \in \mathcal{H}} |X_B| - 2|V(\mathcal{H})|) + (|V| - |V(\mathcal{H})|) + 2|V| \geq 2|V| \).

To see the last inequality, let \( v \in V(\mathcal{H}) \). Then \( v \in V \) and hence \( n - |V(\mathcal{H})| \geq 0 \). If \( v \) belongs to exactly one \( X' \in \mathcal{H} \), then \( v \in X'_I \); so \( v \) contributes 2 in \( \sum_{X \in \mathcal{H}} 2|X_I| \). If \( v \) belongs to at least two \( X', X'' \in \mathcal{H} \), then \( v \in X'_B \) and \( v \in X''_B \); so \( v \) contributes at least 2 in \( \sum_{X \in \mathcal{H}} |X_B| \) and hence \( \sum_{X \in \mathcal{H}} 2|X_I| + \sum_{X \in \mathcal{H}} |X_B| - 2|V(\mathcal{H})| \geq 0 \).

Hence \( 2|V| - 3 \geq r_{\mathcal{R}}(E) \geq 2|V| \), a contradiction.

**Proof of Theorem 2.** Note that in the lower bound on \( |E \setminus F| \), \( d_{G - X_B}(X_I) \geq 6 - |X_B| \) can be replaced by \( d_{G - X_B}(X_I) \geq 6 - 2|X_B| \), and the same proof works. This means that instead of 6-connectivity, we used in fact (6,2)-connectivity.

**Proof of Theorem 4.** Suppose that there exist integers \( k, \ell \) and a graph \( G = (V, E) \) contradicting the theorem. We use the matroid \( \mathcal{M}_{k, \ell} \) defined above. Choose \( F \) a smallest-size set of edges that minimizes the right hand side of (4).

By (2), we can define \( \mathcal{H} \) a collection of subsets of \( V \) such that \( E(X) \cap F \notin \mathcal{H} \) partitions \( F \) and \( r_{\mathcal{R}}(F) = \sum_{X \in \mathcal{H}} (2|X| - 3) \). Since \( G \) is a counterexample and by (2) and (3),

\[
k(2n - 3) + \ell(n - 1) > r_{\mathcal{M}_{k, \ell}}(E) = k \sum_{X \in \mathcal{H}} (2|X| - 3) + \ell(n - c(G_F)) + |E \setminus F| \quad (5)
\]

By \( k \geq 1 \), \( G \) is connected, thus, by (5), \( V \notin \mathcal{H} \). Recall the notations, for \( X \in \mathcal{H} \), \( X_B = X \cap (\bigcup_{Y \in \mathcal{H} \setminus X} Y) \) and \( X_I = X \setminus X_B \) and the definition \( \mathcal{H}' = \{X \in \mathcal{H} : X_I \neq \emptyset\} \). Denote \( \mathcal{K} \) the set of connected components of \( G_F \) intersecting no
set of $H'$. By Remark 1, for $X \in H'$, $X$ induces a connected subgraph of $G_F$, thus a connected component of $G_F$ intersecting $X \in H'$ contains $X$ and is the only connected component of $G_F$ containing $X$. So by definition of $K$,

\[ |\mathcal{H}'| \geq c(G_F) - |\mathcal{K}|. \tag{6} \]

Let us first show a lower bound on $|E \setminus F|$.

**Claim 1.** $|E \setminus F| \geq k \left( 3|\mathcal{H}'| - \sum_{X \in \mathcal{H}'} |X_B| + 3|\mathcal{K}| \right) + \ell c(G_F)$.

**Proof.** For $X \in \mathcal{H}$, $X_I \neq \emptyset$ and $X_I \cup X_B = X \neq V$. Thus by $(6k + 2\ell, 2k)$-connectivity of $G$, for $X \in \mathcal{H}'$ and for $K \in \mathcal{K}$,

\[
\begin{align*}
    d_{G-X_B}(X_I) &\geq (6k + 2\ell) - 2k|X_B|, \tag{7} \\
    d_G(K) &\geq 6k + 2\ell. \tag{8}
\end{align*}
\]

Since every edge of $F$ is induced by an element of $H$ and by definition of $X_I$, for $X \in \mathcal{H}'$, no edge of $F'$ contributes to $d_{G-X_B}(X_I)$. Each $K \in \mathcal{K}$ is a connected component of the graph $G_F$, thus no edge of $F'$ contributes to $d_G(K)$. Hence, by (7), (8), (6) and $\ell \geq 0$, we obtain the required lower bound on $|E \setminus F|$,

\[
|E \setminus F| \geq \frac{1}{2} \left( \sum_{X \in \mathcal{H}'} d_{G-X_B}(X_I) + \sum_{K \in \mathcal{K}} d_G(K) \right) \\
\geq \frac{1}{2} \left( (6k + 2\ell)|\mathcal{H}'| - 2k \sum_{X \in \mathcal{H}'} |X_B| + (6k + 2\ell)|\mathcal{K}| \right) \\
\geq k \left( 3|\mathcal{H}'| - \sum_{X \in \mathcal{H}'} |X_B| + 3|\mathcal{K}| \right) + \ell \left( |\mathcal{H}'| + |\mathcal{K}| \right) \\
\geq k \left( 3|\mathcal{H}'| - \sum_{X \in \mathcal{H}'} |X_B| + 3|\mathcal{K}| \right) + \ell c(G_F). \tag{9}
\]

**Claim 2.** $\sum_{X \in \mathcal{H}' \setminus \mathcal{H}'} |X_B| \geq 3(|\mathcal{H}|-|\mathcal{H}'|)$.

**Proof.** By definition of $\mathcal{H}'$, $X_B = X$ for all $X \in \mathcal{H} \setminus \mathcal{H}'$. So to prove the claim it suffices to show that every $X \in \mathcal{H}$ satisfies $|X| \geq 3$. Suppose there exists $Y \in \mathcal{H}$ such that $|Y| = 2$. By Remark 1 and since $G$ is simple, $Y$ induces exactly one edge $e$. Define $F'' = F - e$ and $\mathcal{H}'' = \mathcal{H} - Y$. Note that $\{E(X) \cap F'', X \in \mathcal{H}'\}$ partitions $F''$, hence by (2) and the choice of $\mathcal{H}$,

\[
\ell r_C(F'') \leq \sum_{X \in \mathcal{H}''} (2|X| - 3) = r_C(F) - (2|Y| - 3) - 3 = r_C(F) - 1. \tag{9}
\]

Note also that $c(G_{F''}) \geq c(G_F)$, thus by (3) and $\ell \geq 0$,

\[
\ell r_C(F'') \leq \ell r_C(F). \tag{10}
\]

Since $|F''| < |F|$, the choice of $F$ implies that $F''$ doesn’t minimizes the right hand side of (4). Hence by (9), (10), the definition of $F''$, $|Y| = 2$, and $k \geq 1$,
we have the following contradiction:

\[
0 < \left( kr_R(F'') + \ell r_C(F'') + |E \setminus F''| \right) - \left( kr_R(F) + \ell r_C(F) + |E \setminus F| \right)
\]

\[
= k \left( r_R(F'') - r_R(F) \right) + \ell \left( r_C(F'') - r_C(F) \right) + \left( |E \setminus F''| - |E \setminus F| \right)
\]

\[
\leq -k + 0 + |\{e\}|
\]

\[
\leq 0.
\]

To finish the proof we show the following inequality with a simple counting argument.

**Claim 3.** \(2|K| + \sum_{X \in \mathcal{H}} 2|X_I| + \sum_{X \in \mathcal{H}} |X_B| \geq 2n.\)

**Proof.** Let \(v \in V.\) If \(v\) belongs to no \(X \in \mathcal{H},\) then \(\{v\} \in K\) and \(v\) contributes \(2\) in \(2|K|\). If \(v\) belongs to exactly one \(X' \in \mathcal{H},\) then \(v \in X'_I\) and \(v\) contributes \(2\) in \(\sum_{X \in \mathcal{H}} 2|X_I|\). If \(v\) belongs to at least two \(X', X'' \in \mathcal{H},\) then \(v \in X'_I, v \in X''_I\) and \(v\) contributes at least \(2\) in \(\sum_{X \in \mathcal{H}} |X_B|\). The claim follows.  

Thus we get, by Claims 1, 2 and 3,

\[
k \sum_{X \in \mathcal{H}} \left( (2|X| - 3) + |E \setminus F| + \ell(n - c(G_F)) \right)
\]

\[
\geq k \sum_{X \in \mathcal{H}} 2|X| - 3k|\mathcal{H}| + k \left( 3|\mathcal{H}'| - \sum_{X \in \mathcal{H}'} |X_B| + 3|K| \right) + \ell c(G_F) + \ell(n - c(G_F))
\]

\[
\geq k \left( \sum_{X \in \mathcal{H}} 2|X| - 3|\mathcal{H}| + 3|\mathcal{H}'| - \sum_{X \in \mathcal{H}'} |X_B| + 3|K| \right) + \ell n
\]

\[
\geq k \left( \sum_{X \in \mathcal{H}} 2|X| - \sum_{X \in \mathcal{H}} |X_B| + 2|\mathcal{K}| \right) + \ell n
\]

\[
\geq k \left( 2|\mathcal{K}| + \sum_{X \in \mathcal{H}} 2|X_I| + \sum_{X \in \mathcal{H}} |X_B| \right) + \ell n
\]

\[
\geq 2kn + \ell n.
\]

By \(k \geq 1\) and \(\ell \geq 0,\) this contradicts (5).

Remark that the proof actually shows that if \(G\) is simple and \((6k + 2\ell, 2k)\)-connected and if \(F \subseteq E\) is such that \(|F| \leq 3k + \ell,\) then \(G' = (V, E \setminus F)\) contains \(k\) rigid spanning subgraphs and \(\ell\) spanning trees pairwise edge disjoint.

**References**


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