Les cahiers Leibniz

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Louis Esperet, Laetitia Lemoine, Frédéric Maffray, Grégory Morel

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Laboratoire G-SCOP
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The chromatic number of $\{P_5, K_4\}$-free graphs

Louis Esperet* Laetitia Lemoine† Frédéric Maffray‡
Grégory Morel§

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Abstract

Gyárfás conjectured that for any tree T every T-free graph G with maximum clique size $\omega(G)$ is $f_T(\omega(G))$-colorable, for some function f_T that depends only on T and $\omega(G)$. Moreover, he proved the conjecture when T is the path P_k on k vertices. In the case of P_5, the best values or bounds known so far are $f_{P_5}(2) = 3$ and $f_{P_5}(q) \leq 3^{q-1}$. We prove here that $f_{P_5}(3) = 5$.

1 Introduction

All graphs considered here are finite and have no loops or multiple edges. The chromatic number of a graph G is denoted by $\chi(G)$ and the clique number by $\omega(G)$. Given a set $\mathcal{F}$ of graphs, a graph G is *$\mathcal{F}$-free* if G has no induced subgraph that is isomorphic to a member of $\mathcal{F}$. A *k-hole* in a graph is an induced cycle of length k, and a *k-antihole* is an induced subgraph isomorphic to the complement of a cycle of length k. We let K_n and P_n respectively denote the complete graph on n vertices and the path on n vertices. Gyárfás [9] conjectured that if T is any tree (or forest) then there is a function f_T such that every T-free graph G satisfies $\chi(G) \leq f_T(\omega(G))$, and he proved the conjecture when T is a path. Gravier, Hoàng and Maffray [8] improved Gyárfás's bound slightly by proving that every P_k-free graph G satisfies $\chi(G) \leq (k-2)^{\omega(G)-1}$. One may wonder whether this exponential bound can be improved. In particular, is there a polynomial function f_k such that every P_k-free graph G satisfies $\chi(G) \leq f_k(\omega(G))$? A positive answer to this question would also imply that a famous conjecture due to Erdős and Hajnal [6] holds true for P_k-free graphs.

*CNRS/Grenoble-INP/UJF-Grenoble 1, G-SCOP (UMR5272), Grenoble, France.
†École Normale Supérieure, Lyon, France.
‡CNRS/Grenoble-INP/UJF-Grenoble 1, G-SCOP (UMR5272), Grenoble, France.
§Projet Mascotte, I3S (CNRS, UNSA) and INRIA, Sophia-Antipolis, France.

This conjecture of Erdős and Hajnal states that for every graph H there exists a constant $\delta(H)$ such that every H-free graph with n vertices contains a clique or a stable set of size $n^{\delta(H)}$. The conjecture is proved only for very few (and small) instance of H. See [4] for a survey. We propose here a first step in the exploration of the question above. When $k = 5$ and $\omega(G) = 3$ (i.e., when G is $\{P_5, K_4\}$-free), the results mentioned above give $\chi(G) \leq 9$. Our main result is the following theorem.

Theorem 1.1. *Let G be a $\{P_5, K_4\}$-free graph. Then $\chi(G) \leq 5$.*

We observe that there exist $\{P_5, K_4\}$-free graphs with chromatic number equal to 5. For example, let H be the graph obtained from the union of two vertex-disjoint 5-holes A and B by adding three vertices x, y, z such that the neighborhood of x is $V(A) \cup V(B)$, the neighborhood of y is $V(A) \cup \{z\}$ and the neighborhood of z is $V(B) \cup \{y\}$. It is easy to check that H is P_5-free and K_4-free and that $\chi(H) = 5$.

In our proof of Theorem 1.1 we will use the following result, which covers the case when our graph contains no odd hole. In Section 7, we show how a proof of our Theorem 1.1 can be obtained without using Theorem 1.2.

Theorem 1.2 (Chudnovsky, Robertson, Seymour and Thomas [5])**.** *Every graph with no odd hole and no K_4 is 4-colorable.*

Proof of Theorem 1.1. Graph G contains no k-hole with $k \geq 6$ (because G contains no P_5) and no ℓ-antihole with $\ell \geq 8$ (because G contains no K_4). If G also contains no 5-hole, then G does not contain any odd hole, and Theorem 1.2 implies that G is 4-colorable. If G contains a 5-hole, the result follows from our Theorem 6.1 in Section 6. □

The proof of Theorem 6.1 itself relies on a sequence of partial results. For this purpose we need to consider four special graphs, which we call the *double diamond*, *simple diamond*, *sapphire* and *ruby*. See Figure 1.

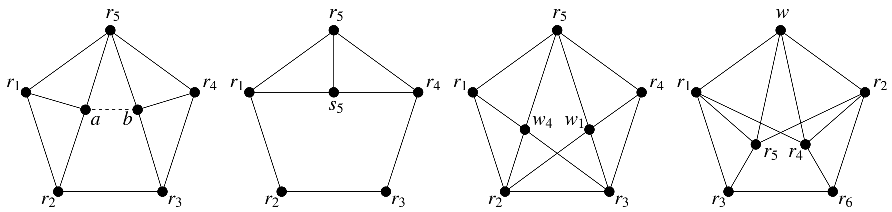

Figure 1: Double diamond, simple diamond, sapphire, ruby.

Let G be any $\{P_5, K_4\}$-free graph. We will prove that:

- If G contains a double diamond, then G is 5-colorable (Theorem 4.2);

- If G contains a simple diamond and no double diamond, then G is 5-colorable (Theorem 4.4);
- If G contains a sapphire and no simple diamond, then G is 5-colorable (Theorem 5.2);
- If G contains a ruby, no simple diamond and no sapphire, then G is 5-colorable (Theorem 5.4);
- If G contains a 5-hole and no simple diamond, no sapphire and no ruby, then G is 5-colorable (Theorem 6.1).

For standard, undefined terms of Graph Theory, we refer to [1]. Here are some additional definitions and notation. In a graph G, a vertex x is *complete* to a subset S of $V(G)$ if x is adjacent to every vertex in S, and x is *anticomplete* to S if x has no neighbor in S. A subset of vertices A is *complete* to a subset B if every vertex of A is complete to B, and A is *anticomplete* to B if every vertex of A is anticomplete to B. A subset S of vertices is *homogeneous* if every vertex of $V(G) \setminus S$ is either complete or anticomplete to S. A *stable set* in a graph G is any subset of pairwise non-adjacent vertices. A graph G *contains* a graph F if F is isomorphic to an induced subgraph of G.

The class of P_5-free graphs is of particular interest in graph theory. It generalizes many classes, such as *split graphs*, *cographs*, *$2K_2$-free graphs*, *P_4-sparse graphs*, etc. It is the smallest class (by inclusion) defined by only one connected forbidden induced subgraph for which the complexity status of the Maximum Independent Set problem is still unknown, despite much research; see e.g. [12]. On the positive side it is known that, that, for fixed k, one can decide in polynomial time if a P_5-graph is k-colorable [11]. More structural results on P_5-free graphs can be found in particular in [2] and [3].

2 $\{P_5, K_3\}$-free graphs

Before considering $\{P_5, K_4\}$-graphs, it is convenient to know the structure of $\{P_5, K_3\}$-free graphs. Let us call 5-*ring* any graph whose vertex-set can be partitioned into five non-empty stable sets $R_1, \dots, R_5$ such that for each i (with subscripts modulo 5) R_i is complete to $R_{i-1} \cup R_{i+1}$ and anticomplete to $R_{i-2} \cup R_{i+2}$. The following result is due to Sumner [13]. We include a proof for the sake of completeness.

Theorem 2.1 ([13]). *Let G be a $\{P_5, K_3\}$-free graph. Then each component of G is either bipartite or a 5-ring.*

Proof. Suppose that a component D of G is not bipartite; so D contains an odd cycle. Since G is K_3-free and P_5-free, D contains a cycle of length 5, so D contains a 5-ring $R = (R_1, \dots, R_5)$, and we may assume that R is the largest such subgraph of D. For each $i \in \{1, \dots, 5\}$, pick a vertex r_i in R_i. Suppose that $D \neq R$. Since D is connected, some vertex x of $D \setminus R$ has a neighbor u_1 in R, say $u_1 \in R_1$. Then x has no neighbor u in $R_2 \cup R_5$, for otherwise $\{x, u_1, u\}$ induces a K_3. By the same argument (applied to $u_3 \in R_3$ and $u_4 \in R_4$), and up to symmetry, we may assume that x has no neighbor in R_3. If x has a non-neighbor v in R_4, then x-u_1-r_2-r_3-v is an induced P_5. So x is complete to R_4, and by the same argument it is complete to R_1. Thus $(R_1, R_2, R_3, R_4, R_5 \cup \{x\})$ is a 5-ring, which contradicts the choice of R. So $D = R$ and D is a 5-ring. $\square$

It follows easily from the preceding theorem that every $\{P_5, K_3\}$-free graph is 3-colorable. We will need a stronger statement, as follows.

Corollary 2.2. *Let G be a $\{P_5, K_3\}$-free graph. Let S and T be disjoint stable sets in G such that every vertex of T has a neighbor in S (T may be empty). Then G admits a 3-coloring where all vertices of S have color 1 and all vertices of T have color 2.*

Proof. We may assume that G is connected, for otherwise it suffices to consider each component of G separately.
First suppose that G is bipartite, with bipartition (A, B). Suppose that there are vertices a, b of T with $a \in A$ and $b \in B$. By definition of T, there are vertices u, v of S with $au, bv \in E$, whence $u \in B$ and $v \in A$. Since G is connected, there is a shortest path P between $\{a, u\}$ and $\{b, v\}$, and it is easy to check that the subgraph induced by $V(P) \cup \{a, b, u, v\}$ contains a P_5, a contradiction. Therefore we may assume, up to symmetry, that B contains no vertex from T, so $T \subseteq A \setminus S$. We assign color 1 to all vertices of S, color 2 to all vertices of $A \setminus S$, and color 3 to all vertices of $B \setminus S$. This coloring satisfies the requirement.
Now we may assume, by Theorem 2.1, that G is a 5-ring $(R_1, \dots, R_5)$, and, up to symmetry, that S has a vertex in R_1. So S has no vertex in $R_2 \cup R_5$. If S also has no vertex in $R_3 \cup R_4$, then (because every vertex of T has a neighbor in S) we have $T \subseteq R_2 \cup R_5$. Otherwise, up to symmetry, S has a vertex in R_3 and none in R_4 and up to symmetry again T has no vertex in R_4. It follows that $T \subseteq R_2 \cup R_5$ also in this case. In either case, we assign color 1 to all vertices of $R_1 \cup R_3$, color 2 to all vertices of $R_2 \cup R_5$, and color 3 to all vertices of R_4. This coloring satisfies the requirement. $\square$

3 Neighbors of a 5-hole

In a graph G, given a 5-hole with vertex-set C, let us say that a vertex x in $V(G) \setminus C$ is of *type 0* (on C) if it has no neighbor in C, of *type i* (for any i in $\{1, \dots, 5\}$), if x has exactly i neighbors on C and these neighbors are consecutive, of *type 2t* if it has exactly two neighbors on C and they are not consecutive, and of *type 3y* if it has exactly three neighbors and they are not consecutive (i.e., two of them are adjacent and the third one is not adjacent to the first two). Clearly, these types cover all possibilities, in other words, every vertex of

$V(G) \setminus C$ is of type 0, 1, 2, 3, 4, 5, 2t or 3y on C. In a P_5-free graph one can say a little more, as follows.

Lemma 3.1. *Let G be a P_5-free graph that contains a 5-hole C. Then every vertex of $V(G) \setminus C$ is of type 0, 2t, 3, 3y, 4 or 5 on C.*

Proof. If a vertex x is of type 1 or 2 on C, then it is easy to see that $G[C \cup \{x\}]$ contains a P_5 (of which x is one endvertex). □

In the next sections, we deal with the case when G has a 5-hole with neighbors of type 3 (i.e., G contains a double diamond or a simple diamond), then neighbors of type 4 (i.e., G contains a sapphire or a ruby). Then we consider the case when G has a 5-hole with all remaining types of neighbors. The following lemma will be used many times.

Lemma 3.2. *Let G be a connected $\{P_5, K_4\}$-free graph. Let D be a connected induced subgraph of G that contains a K_3, and let $Z = \{x \in V(G) \mid x$ has no neighbor in $D\}$. Then every component of $G[Z]$ is homogeneous, and $G[Z]$ is 3-colorable.*

Proof. Suppose that some component W of Z is not homogeneous. So there exist a vertex x in $V(G) \setminus W$ and adjacent vertices u and v in W such that x is adjacent to u and not to v. By the definition of Z, x in not in D and has a neighbor in D. Vertex x is not complete to D, for otherwise, since D contains a K_3, G would contain a K_4. So, because D is connected, there are adjacent vertices a and b in D such that x is adjacent to a and not to b. But then b-a-x-u-v is a P_5. So W is homogeneous. Since G is connected, there is a vertex x in $V(G) \setminus W$ that is adjacent to W, and by the preceding point x is complete to W. It follows that $G[W]$ contains no K_3, for otherwise, adding x, G would contain a K_4. By Theorem 2.1, $G[W]$ is 3-colorable. Since this holds for every component W of Z, we obtain that $G[Z]$ is 3-colorable. □

4 5-holes with neighbors of type 3

4.1 Double diamonds

A *double diamond* is a graph with seven vertices $r_1, \dots, r_5, a, b$ such that $r_1, \dots, r_5$ induce a 5-hole with edges $r_i r_{i+1}$ (modulo 5), vertex a is adjacent to r_1, r_2 and r_5 and not adjacent to r_3 and r_4, and vertex b is adjacent to r_3, r_4 and r_5 and not adjacent to r_1 and r_2. Vertices a and b may be adjacent or not. See Figure 1.

Lemma 4.1. *Let G be a connected $\{P_5, K_4\}$-free graph. Suppose that G contains a double diamond, with vertex-set $D = \{r_1, \dots, r_5, a, b\}$ and edges as above. Let:*
- $R_i = \{x \in V(G) \mid N_D(x) = N_D(r_i)\}$ for each i in $\{2, 3, 5\}$,
- $R_1 = \{x \in V(G) \mid x$ is complete to $\{r_2, r_5\}$ and anticomplete to $\{r_3, r_4\}\}$,

- $R_4 = \{x \in V(G) \mid x$ *is complete to* $\{r_3, r_5\}$ *and anticomplete to* $\{r_1, r_2\}\}$,
- $Y = \{x \in V(G) \mid N_D(x) = \{r_2, r_3, r_5\}\}$,
- $F = \{x \in V(G) \mid N_D(x) = \{r_1, r_2, r_3, r_4\}$ *or* $\{r_1, r_2, r_3, r_4, r_5\}\}$,
- $Z = \{x \in V(G) \mid N_D(x) = \emptyset\}$.

Then $V(G) = R_1 \cup R_2 \cup R_3 \cup R_4 \cup R_5 \cup Y \cup F \cup Z$.
Moreover, if ab *is not an edge, then* $F = \emptyset$*, and if* ab *is an edge, then* $Y = \emptyset$.

Proof. Note that $r_i \in R_i$ for each i in $\{1, \ldots, 5\}$, $a \in R_1$ and $b \in R_4$. Now consider any vertex x of $V(G) \setminus D$. Let $C = \{r_1, \ldots, r_5\}$. So C induces a 5-hole. By Lemma 3.1, x is of type 0, 2t, 3, 3y, 4, or 5 on C. Let us analyze each case. If x is of type 0 on C, then x is not adjacent to a, for otherwise x-a-r_2-r_3-r_4 is a P_5, and similarly x is not adjacent to b. So $x \in Z$.
Now suppose that x is of type 2t or 3 on C. So $N_C(x)$ is equal to $\{r_{j-1}, r_{j+1}\}$ or $\{r_{j-1}, r_j, r_{j+1}\}$ for some j in $\{1, \ldots, 5\}$. If $j \in \{1, 4\}$, then x is in R_j. If $j = 2$, then xa is an edge, for otherwise a-r_1-x-r_3-r_4 is a P_5, xr_2 is not an edge, for otherwise $\{x, r_1, a, r_2\}$ induces a K_4, and xb is not an edge, for otherwise r_2-r_1-x-b-r_4 is a P_5. So $x \in R_2$. Likewise, if $j = 3$, then $x \in R_3$. If $j = 5$, then xa is an edge, for otherwise a-r_2-r_3-r_4-x is a P_5, and similarly xb is an edge, and xr_5 is not an edge, for otherwise $\{x, r_1, a, r_5\}$ induces a K_4. So $x \in R_5$.
Now suppose that x is of type 3y on C. So $N_C(x)$ is equal to $\{r_{j-2}, r_j, r_{j+2}\}$ for some j in $\{1, \ldots, 5\}$. If $j = 1$, then either $\{x, b, r_3, r_4\}$ induces a K_4 (if xb is an edge) or b-r_4-x-r_1-r_2 is a P_5 (if xb is not an edge), a contradiction; thus $j \neq 1$. Similarly, $j \neq 4$. If $j = 2$, then either $\{x, b, r_4, r_5\}$ induces a K_4 (if xb is an edge) or b-r_4-x-r_2-r_1 is a P_5 (if xb is not an edge), a contradiction; thus $j \neq 2$. Similarly, $j \neq 3$. So $j = 5$. Then xa is not an edge, for otherwise r_1-a-x-r_3-r_4 is a P_5, and similarly xb is not an edge. So $x \in Y$. Moreover, ab is not an edge, for otherwise x-r_2-a-b-r_4 is a P_5. This shows that when ab is an edge, Y must be empty.
Finally, suppose that x is of type 4 or 5 on C. If x is not adjacent to r_j for some j in $\{1, 2\}$ (and so x is of type 4), then xb is not an edge, for otherwise $\{x, b, r_3, r_4\}$ induces a K_4; but then either b-r_4-x-r_2-r_1 or b-r_4-x-r_1-r_2 is a P_5. So x is adjacent to r_1 and r_2, and similarly it is adjacent to r_3 and r_4. Then xa is not an edge, for otherwise $\{x, a, r_1, r_2\}$ induces a K_4; and similarly xb is not an edge. So $x \in F$. Moreover, ab is an edge, for otherwise a-r_1-x-r_4-b is a P_5. This shows that when ab is not an edge, F must be empty. This completes the proof of the lemma. □

Theorem 4.2. *Let* G *be a* $\{P_5, K_4\}$*-free graph that contains a double diamond. Then* G *is* 5*-colorable.*

Proof. Let $D = \{r_1, \ldots, r_5, a, b\}$ be the vertex-set of a double diamond in G, with the same notation as above. Let sets $R_1, \ldots, R_5, Y, F$ and Z be defined as in Lemma 4.1. We observe that:

$$\text{Each of } R_2 \cup R_5,\ R_3 \cup R_5 \text{ and } Y \cup F \text{ is a stable set.} \tag{1}$$

Suppose that there are adjacent vertices u and v in any of these three sets. If

$u, v \in R_2 \cup R_5$, then $\{u, v, a, r_1\}$ induces a K_4. If $u, v \in R_3 \cup R_5$, then $\{u, v, b, r_4\}$ induces a K_4. If $u, v \in Y \cup F$, then $\{u, v, r_2, r_3\}$ induces a K_4. Thus (1) holds.

$$N(Z) \subseteq Y, \text{ and every component of } Z \text{ is homogeneous.} \tag{2}$$

The fact that every component of Z is homogeneous follows from Lemma 3.2. Now suppose that there is an edge zu with $z \in Z$ and $u \in V(G) \setminus (Z \cup Y)$. By Lemma 4.1, we have $u \in R_i \cup F$ for some i in $\{1, \dots, 5\}$. If $u \in R_i$, then z-u-r_{i+1}-r_{i+2}-r_{i+3} is a P_5. If $u \in F$, then ab is an edge by Lemma 4.1, and then a-b-r_4-u-z is a P_5. So $N(Z) \subseteq Y$. Thus (2) holds.

Now we build a 5-coloring of G.

First suppose that ab is not an edge. By Lemma 4.1, $F = \emptyset$. Every vertex of $R_1 \cup R_4 \cup Y$ is adjacent to r_5, so the subgraph $G[R_1 \cup R_4 \cup Y]$ contains no K_3, and so, by Theorem 2.1, it is 3-colorable. We assign colors 1, 2 and 3 to its vertices. By Corollary 2.2, we can ensure that all vertices of Y receive color 1. We assign color 4 to the vertices of R_2, and color 5 to the vertices of $R_3 \cup R_5$. By (1), this yields a 5-coloring of $G \setminus Z$. By Lemma 3.2, $G[Z]$ is 3-colorable. We assign colors 2, 3 and 4 to the vertices of Z. (Recall that all vertices of Y have color 1.) Thus we obtain a 5-coloring of G.

Now suppose that ab is an edge. By Lemma 4.1, $Y = \emptyset$. By (2) and since $Y = \emptyset$ and G is connected, we have $Z = \emptyset$. Therefore $V(G) = R_1 \cup \ldots \cup R_5 \cup F$. We claim that every vertex x in $R_1 \cup R_4$ is adjacent to exactly one of a and b. For suppose the contrary. Up to symmetry, we can assume $x \in R_1$. If x is adjacent to both a and b, then $\{x, a, b, r_5\}$ induces a K_4. If x is adjacent to none of a and b, then x-r_2-a-b-r_4 is a P_5, a contradiction. So the claim holds. For each u in $\{a, b\}$, let $R_u = \{x \in R_1 \cup R_4 \mid x \text{ is adjacent to } u\}$. So $R_1 \cup R_4 = R_a \cup R_b$. We observe that R_a is a stable set, for if it contained two adjacent vertices x and y then $\{x, y, a, r_5\}$ would induce a K_4. Likewise, R_b is a stable set. It follows that R_a, R_b, R_2, $R_3 \cup R_5$ and F form a 5-coloring of G. $\square$

4.2 Simple diamonds

A *simple diamond* is a graph with six vertices $r_1, \dots, r_5$ and s_5 such that $r_1, \dots, r_5$ induce a 5-hole with edges $r_i r_{i+1}$ (modulo 5) and the neighbors of s_5 are r_1, r_4 and r_5. See Figure 1.

Lemma 4.3. *Let G be a connected $\{P_5, K_4\}$-free graph that contains a simple diamond, with vertex-set $D = \{r_1, \dots, r_5, s_5\}$ and edges as above. Assume that G contains no double diamond. Let:*
- $R_i = \{x \in V(G) \mid N_D(x) = N_D(r_i)\}$ *for each i in* $\{1, 2, 3, 4\}$,
- $R_5 = \{x \in V(G) \mid x$ *is complete to* $\{r_1, r_4\}$ *and anticomplete to* $\{r_2, r_3\}\}$,
- $Y_i = \{x \in V(G) \mid N_D(x) = \{r_i, r_{i-2}, r_{i+2}\}\}$ *for each i in* $\{1, 4\}$,
- $Y_5 = \{x \in V(G) \mid N_D(x) = \{r_2, r_3, r_5, s_5\}\}$,
- $F = \{x \in V(G) \mid \{r_1, r_2, r_3, r_4\} \subseteq N_D(x)\}$,

- $Z = \{x \in V(G) \mid N_D(x) = \emptyset\}$.
Then $V(G) = R_1 \cup R_2 \cup R_3 \cup R_4 \cup R_5 \cup Y_1 \cup Y_4 \cup Y_5 \cup F \cup Z$.

Proof. Note that $r_i \in R_i$ for each i in $\{1, \dots, 5\}$ and $s_5 \in R_5$. Now consider any vertex x of $V(G) \setminus D$. Let $C = \{r_1, \dots, r_5\}$. So C induces a 5-hole. By Lemma 3.1, x is of type 0, 2t, 3, 3y, 4 or 5 on C. Let us analyze each case.
If x is of type 0 on C, then x is not adjacent to s_5, for otherwise x-s_5-r_1-r_2-r_3 is a P_5. Thus $x \in Z$.
Now suppose that x is of type 2t or 3 on C. So $N_C(x)$ is equal to $\{r_{j-1}, r_{j+1}\}$ or $\{r_{j-1}, r_j, r_{j+1}\}$ for some j in $\{1, \dots, 5\}$. If $j = 1$, then xs_5 is an edge, for otherwise x-r_2-r_3-r_4-s_5 is a P_5. Thus $x \in R_1$. Likewise, if $j = 4$, then $x \in R_4$. Now let $j = 2$. If xs_5 is an edge, then xr_2 is an edge, for otherwise r_2-r_3-x-s_5-r_5 is a P_5; but then $D \cup \{x\}$ induces a double diamond. So xs_5 is not an edge, and $x \in R_2$. Likewise, if $j = 3$, then $x \in R_3$. If $j = 5$, then $x \in R_5$.
Now suppose that x is of type 3y on C. So $N_C(x)$ is equal to $\{r_{j-2}, r_j, r_{j+2}\}$ for some j in $\{1, \dots, 5\}$. If $j = 1$, then xs_5 is not an edge, for otherwise r_2-r_3-x-s_5-r_5 is a P_5; so $x \in Y_1$. Likewise, if $j = 4$, then $x \in Y_4$. If $j = 2$, then xs_5 is not an edge, for otherwise $\{x, r_4, r_5, s_5\}$ induces a K_4; but then r_3-r_2-x-r_5-s_5 is a P_5. So $j \neq 2$, and similarly $j \neq 3$. If $j = 5$, then xs_5 is an edge, for otherwise x-r_3-r_4-s_5-r_1 is a P_5; and so $x \in Y_5$.
Finally, suppose that x is of type 4 or 5 on C. If x is not adjacent to r_j for some j in $\{1, 2\}$ (and so x is of type 4), then xs_5 is not an edge, for otherwise $\{x, r_4, r_5, s_5\}$ induces a K_4; but then either $D \cup \{x\}$ induces a double diamond (when $j = 1$) or r_2-r_3-x-s_5-r_5 is a P_5 (when $j = 2$). So x is adjacent to r_1 and r_2, and similarly it is adjacent to r_3 and r_4. Thus $x \in F$. □

Theorem 4.4. *Let G be a $\{P_5, K_4\}$-free graph that contains a simple diamond. Then G is 5-colorable.*

Proof. By Theorem 4.2, we may assume that G contains no double diamond. We may also assume that G is connected. Let $D = \{r_1, \dots, r_5, s_5\}$ be the vertex-set of a simple diamond in G, with the same notation as above. Let sets $R_1, \dots, R_5, Y_1, Y_4, Y_5, F$ and Z be defined as in Lemma 4.3. We observe that:

$$R_2,\ R_3,\ R_1 \cup R_4 \cup Y_5,\ F \cup Y_1 \text{ and } F \cup Y_4 \text{ are stable sets.} \tag{3}$$

Suppose that there are two adjacent vertices u and v in any of these five sets. If $u, v \in R_2$, then s_5 is adjacent to at most one of u and v, for otherwise $\{u, v, r_1, s_5\}$ induces a K_4; but then $(D \setminus \{r_2\}) \cup \{u, v\}$ induces a double diamond, a contradiction. The proof is similar for R_3. If $u, v \in R_1 \cup R_4 \cup Y_5$, then $\{u, v, r_5, s_5\}$ induces a K_4. If $u, v \in F \cup Y_1$, then $\{u, v, r_3, r_4\}$ induces a K_4. The proof is similar for $F \cup Y_4$. Thus (3) holds.

$$Y_5 \text{ is complete to } Y_1 \cup Y_4. \tag{4}$$

Suppose that there are non-adjacent vertices u and v with $u \in Y_5$ and $v \in Y_1 \cup Y_4$. Up to symmetry, let $v \in Y_1$. Then u-r_2-r_1-v-r_4 is a P_5. Thus (4) holds.

$$N(Z) \subseteq Y_1 \cup Y_4 \cup Y_5 \cup F. \tag{5}$$

Suppose that there is an edge zu with $z \in Z$ and $u \in V(G) \setminus (Z \cup Y_1 \cup Y_4 \cup Y_5 \cup F)$. By Lemma 4.3, we have $u \in R_i$ for some i in $\{1, \dots, 5\}$. But then z-u-r_{i+1}-r_{i+2}-r_{i+3} is a P_5. Thus (5) holds.

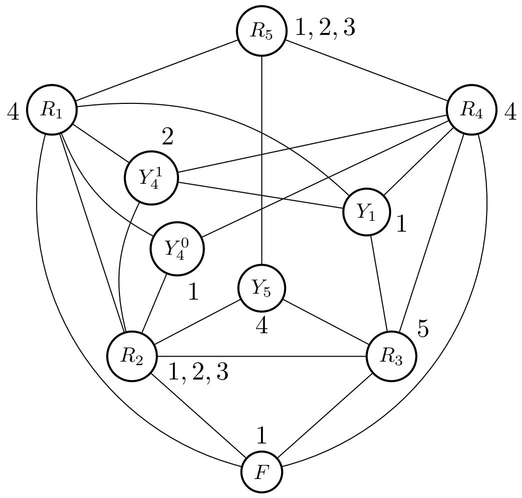

Figure 2: A 5-coloring of $G \setminus Z$ when G contains a simple diamond.

Now we build a 5-coloring of $G \setminus Z$. Every vertex of $R_2 \cup R_5 \cup Y_1 \cup Y_4 \cup F$ is adjacent to r_1, so the induced subgraph $G[R_2 \cup R_5 \cup Y_1 \cup Y_4 \cup F]$ contains no K_3, and so, by Theorem 2.1, it is 3-colorable. Let $Y_4^1 = \{y \in Y_4 \mid y \text{ has a neighbor in } Y_1\}$ and $Y_4^0 = Y_4 \setminus Y_4^1$. Let $S = F \cup Y_1 \cup Y_4^0$ and $T = Y_4^1$. By (3), S and T are stable sets, and by their definition, every vertex of T has a neighbor in S. By Corollary 2.2, we can color the vertices of $G[R_2 \cup R_5 \cup Y_1 \cup Y_4 \cup F]$ with three colors 1, 2 and 3 so that all vertices of S receive color 1 and all vertices of T receive color 2. We assign color 4 to the vertices of $R_1 \cup R_4 \cup Y_5$, and color 5 to the vertices of R_3. By (3), this yields a 5-coloring of $G \setminus Z$ (see Figure 2).

Now consider any component X of Z. By Lemma 3.2, X is homogeneous and $G[X]$ is 3-colorable. By (5), every vertex of $N(X)$ has color 1, 2 or 4. If X has only one vertex, we give it color 3. Now suppose that X has at least two vertices, so it contains two adjacent vertices x and x'. We observe that $N(X)$ cannot contain both a vertex t of color 2 and a vertex y of color 4, for otherwise we must have $t \in T$ $(\subseteq Y_4)$ and $y \in Y_5$ and then, by (4), $\{x, x', t, y\}$ induces a K_4. Consequently, X can be colored with colors 3, 5 and one of $2, 4$. Thus we obtain a 5-coloring of G. □

5 5-holes with neighbors of type 4

5.1 Sapphires

A *sapphire* is a graph with seven vertices $r_1, \dots, r_5$, w_1, w_4 such that $r_1, \dots, r_5$ induce a 5-hole, with edges r_ir_{i+1} (modulo 5), the neighborhood of w_1 is $\{r_2, r_3, r_4, r_5\}$ and the neighborhood of w_4 is $\{r_1, r_2, r_3, r_5\}$. See Figure 1.

Lemma 5.1. *Let G be a $\{P_5, K_4\}$-free that contains no simple diamond. Suppose that G contains a sapphire, with vertex-set $S = \{r_1, \dots, r_5, w_1, w_4\}$ and edges as above. Let:*
- $R_i = \{x \in V(G) \mid N_S(x) = N_S(r_i)\}$ for each $i \in \{1, \dots, 5\}$,
- $W_j = \{x \in V(G) \mid N_S(x) = N_S(w_j)\}$ for each $j \in \{1, 4\}$,
- $T_1 = \{x \in V(G) \mid N_S(x) = \{r_2, r_5\}\}$,
- $T_4 = \{x \in V(G) \mid N_S(x) = \{r_3, r_5\}\}$,
- $Y = \{x \in V(G) \mid N_S(x) = \{r_2, r_3, r_5\}\}$.
Then $V(G) = R_1 \cup \dots \cup R_5 \cup W_1 \cup W_4 \cup T_1 \cup T_4 \cup Y$.

Proof. Let $C = \{r_1, \dots, r_5\}$; so C induces a 5-hole. Clearly, every vertex of C is in $R_1 \cup \dots \cup R_5 \cup W_1 \cup W_4$. Let us now consider a vertex x in $V(G) \setminus C$. By Lemma 3.1, x is of type 0, 2t, 3, 3y, 4 or 5 on C.
First suppose that x is of type 4 or 5 on C. If $\{r_1, r_2, r_3, r_4\} \subseteq N(x)$, then x has no neighbor w in $\{w_1, w_4\}$, for otherwise $\{x, w, r_2, r_3\}$ induces a K_4; but then w_1-r_4-x-r_1-w_4 is a P_5. Therefore it must be that x is of type 4 with $N_C(x) = C \setminus \{r_j\}$ for some j in $\{1, \dots, 4\}$. If $j = 1$, then xw_1 is not an edge, for otherwise $\{x, w_1, r_2, r_3\}$ induces a K_4, and xw_4 is not an edge, for otherwise w_1-r_4-x-w_4-r_1 is a P_5. Thus $x \in W_1$. Likewise, if $j = 4$, then $x \in W_4$. If $j = 2$, then xw_1 is not an edge, for otherwise $\{x, w_1, r_4, r_5\}$ induces a K_4, and xw_4 is not an edge, for otherwise $\{x, w_4, r_1, r_5\}$ induces a K_4; but then w_1-r_4-x-r_1-w_4 is a P_5. If $j = 3$ a similar contradiction occurs.
If x is of type 3, then $C \cup \{x\}$ induces a simple diamond, a contradiction.
Now suppose that x is of type 3y. So $N_C(x) = \{r_{j-2}, r_j, r_{j+2}\}$ for some j in $\{1, \dots, 5\}$. If $j = 1$, then xw_1 is not an edge, for otherwise $\{x, w_1, r_3, r_4\}$ induces a K_4, and xw_4 is an edge, for otherwise w_1-r_4-x-r_1-w_4 is a P_5; but then $\{x, r_1, r_2, r_4, w_1, w_4\}$ induces a simple diamond. If $j = 4$ a similar contradiction occurs. If $j = 2$, then xw_1 is not an edge, for otherwise $\{x, w_1, r_4, r_5\}$ induces a K_4, and xw_4 is not an edge, for otherwise r_1-w_4-x-r_4-w_1 is a P_5; but then r_1-w_4-r_3-r_4-x is a P_5. If $j = 3$ a similar contradiction occurs. If $j = 5$, then x is not adjacent to any vertex w in $\{w_1, w_4\}$, for otherwise $\{x, w, r_2, r_3\}$ induces a K_4; thus $x \in Y$.
Now suppose that x is of type 2t. So $N_C(x) = \{r_{j-1}, r_{j+1}\}$ for some j in $\{1, \dots, 5\}$. Let $j = 1$. If x is adjacent to w_4, then it is not adjacent to w_1, for otherwise r_1-w_4-x-w_1-r_4 is a P_5; and so $x \in R_1$. If x is not adjacent to w_4, then it is not adjacent to w_1, for otherwise r_1-w_4-r_3-w_1-x is a P_5; and so $x \in T_1$. Likewise, if $j = 4$, then $x \in R_4 \cup T_4$. If $j = 2$, then xw_1 is an edge, for otherwise x-r_1-r_2-w_1-r_4 is a P_5, and xw_4 is an edge, for otherwise w_4-r_1-x-w_1-r_4 is a P_5.

So $x \in R_2$. Likewise, if $j = 3$, then $x \in R_3$. If $j = 5$, then xw_1 is an edge, for otherwise x-r_1-r_5-w_1-r_3 is a P_5, and by symmetry xw_4 is an edge. So $x \in R_5$. Finally, suppose that x is of type 0. Let Z be the set of vertices that have no neighbor in C. Since G is connected, there is an edge zy with $z \in Z$ and $y \notin Z$. By the preceding arguments, y satisfies the conclusion of the lemma. In either case, we observe that there are three vertices s, s', s'' of S such that s-s'-s'' is a P_3 and y is adjacent to s and neither to s' or s'' (if $y \in R_i$, consider the P_3 r_{i+1}-r_{i+2}-r_{i+3}; if $y \in W_1 \cup T_4$, consider r_3-w_4-r_1; the other cases are symmetric). Then z-y-s-s'-s'' is a P_5. This completes the proof of the lemma. □

Theorem 5.2. *Let G be a $\{P_5, K_4\}$-free graph that contains a sapphire. Then G is 5-colorable.*

Proof. By Theorem 4.4, we may assume that G contains no simple diamond. Let $S = \{r_1, \ldots, r_5, w_1, w_4\}$ be the vertex-set of a sapphire in G, with the same notation as above. Let sets $R_1, \ldots, R_5$, W_1, W_4, T_1, T_4 and Y be defined as in Lemma 5.1. We know that $V(G) = R_1 \cup \cdots \cup R_5 \cup W_1 \cup W_4 \cup T_1 \cup T_4 \cup Y$.

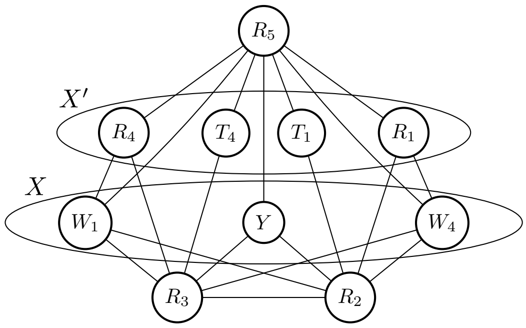

Figure 3: A 5-coloring of a graph that contains a sapphire.

We observe that R_2 is a stable set, because its vertices are all adjacent to r_3 and w_1 (which are adjacent) and G contains no K_4. Likewise, R_3 is a stable set, and R_5 is a stable set, because its vertices are all adjacent to r_1 and w_4. Let $X = W_1 \cup W_4 \cup Y$. Then X is a stable set, because its vertices are all adjacent to r_2 and r_3. Let $X' = R_1 \cup R_4 \cup T_1 \cup T_4$. Suppose that X' contains two adjacent vertices x and y. If x and y are both in $R_1 \cup T_1$, then $\{x, y, r_2, r_3, r_4, r_5\}$ induces a simple diamond. The same holds if x and y are both in $R_4 \cup T_4$. So we may assume that $x \in R_1 \cup T_1$ and $y \in R_4 \cup T_4$. Then xw_4 is an edge, for otherwise x-y-r_3-w_4-r_1 is a P_5. By symmetry, yw_1 is an edge. But then r_1-w_4-x-y-w_1 is a P_5. Thus X' is a stable set. Hence R_2, R_3, R_5, X and X' form a 5-coloring of G (see Figure 3). □

5.2 Rubies

A *ruby* is a graph with seven vertices $r_1, \ldots, r_6, w$ such that $r_1, \ldots, r_6$ induce a 6-antihole, with non-edges $r_i r_{i+1}$ (modulo 6), and the neighborhood of w is $\{r_1, r_2, r_4, r_5\}$. See Figure 1.

Lemma 5.3. *Let G be a connected $\{P_5, K_4\}$-free graph that contains no simple diamond. Suppose that G contains a ruby, with vertex-set $\{r_1, \ldots, r_6, w\}$ and edges as above. Let $C = \{r_1, \ldots, r_6\}$. For each i in $\{1, \ldots, 6\}$, let:*
- $R_i = \{x \in V(G) \mid N_C(x) = N_C(r_i)\}$,
- $D_{i,i+1} = \{x \in V(G) \mid N_C(x) = \{r_i, r_{i+1}\}\}$,
- $F_{i,i+1} = \{x \in V(G) \mid N_C(x) = \{r_{i-1}, r_i, r_{i+1}, r_{i+2}\}\}$,
- $W = \{x \in V(G) \mid N_C(x) = N_C(w)\}$,
- $Z = \{x \in V(G) \mid x$ has no neighbor in $C \cup \{w\}\}$,
where all subscripts are modulo 6. Then $V(G) = \bigcup_{1 \le i \le 6}(R_i \cup D_{i,i+1} \cup F_{i,i+1}) \cup W \cup Z$.

Proof. Let $A = \{w, r_1, r_2, r_3, r_6\}$ and $B = \{w, r_3, r_4, r_5, r_6\}$. Note that each of A and B induces a 5-hole. Consider any vertex x in $V(G) \setminus (C \cup \{w\})$, and let $X = N(x) \cap \{r_1, r_2, r_4, r_5\}$.
First suppose that $|X| = 0$. Then x is not adjacent to r_3, for otherwise x-r_3-r_5-r_2-r_4 is a P_5, and by symmetry it is not adjacent to r_6, and it is not adjacent to w, for otherwise x-w-r_1-r_3-r_6 is a P_5. Thus $x \in Z$.
Now suppose that $|X| = 1$. Up to symmetry, let $X = \{r_1\}$. Then x is adjacent to r_6, for otherwise x-r_1-r_5-r_2-r_6 is a P_5, and x is not adjacent to r_3, for otherwise x-r_3-r_5-r_2-r_4 is a P_5. Thus $x \in D_{6,1}$. The other (symmetric) cases are similar.
Now suppose that $|X| = 2$. First let $X = \{r_1, r_4\}$. Then x is not adjacent to w, for otherwise $\{x, w, r_1, r_4\}$ induces a K_4, x is adjacent to r_3, for otherwise x-r_4-w-r_5-r_3 is a P_5, and x is adjacent to r_6, for otherwise x-r_1-w-r_2-r_6 is a P_5. But then $A \cup \{x\}$ induces a simple diamond. By symmetry, the same contradiction occurs if $X = \{r_2, r_5\}$. Now let $X = \{r_1, r_5\}$. Then x is not adjacent to any vertex u in $\{r_3, w\}$, for otherwise $\{x, u, r_1, r_5\}$ induces a K_4, and x is adjacent to r_6, for otherwise x-r_1-w-r_2-r_6 is a P_5. Thus $x \in R_3$. Likewise, if $X = \{r_2, r_4\}$ then $x \in R_6$. Now let $X = \{r_1, r_2\}$. If x has any neighbor in $\{r_3, r_6, w\}$, then it must have at least two neighbors in that set, including w, for otherwise $G[B \cup \{x\}]$ contains a P_5. Thus $N_C(x)$ is equal to either $\{r_1, r_2\}$ (so $x \in D_{1,2}$) or $\{r_1, r_2, r_3\}$ (so $x \in R_5$) or $\{r_1, r_2, r_6\}$ (so $x \in R_4$) or $\{r_1, r_2, r_3, r_6\}$ (so $x \in F_{1,2}$). If $X = \{r_4, r_5\}$ the conclusion is similar.
Now suppose that $|X| = 3$. Up to symmetry, let $X = \{r_1, r_2, r_4\}$. Then x is not adjacent to any vertex u in $\{r_6, w\}$, for otherwise $\{x, u, r_2, r_4\}$ induces a K_4, and x is adjacent to r_3, for otherwise x-r_4-w-r_5-r_3 is a P_5. Thus $N_C(x) = \{r_1, r_2, r_3, r_4\}$, so $x \in F_{2,3}$. The other (symmetric) cases are similar.
Finally suppose that $|X| = 4$. Then x is not adjacent to any vertex u in $\{r_3, r_6\}$, for otherwise $\{x, u, r_1, r_5\}$ or $\{x, u, r_2, r_4\}$ induces a K_4. Thus $N_C(x) = \{r_1, r_2, r_4, r_5\}$, so $x \in W$. □

Theorem 5.4. *Let G be a $\{P_5, K_4\}$-free graph that contains a ruby, Then G is*

5-*colorable.*

Proof. By Theorem 4.4, we may assume that G contains no simple diamond. Let $R = \{r_1, \ldots, r_6, w\}$ be the vertex-set of a ruby in G, with the same notation as above. By Lemma 5.3, $V(G)$ is the union of the twenty sets $R_1, \ldots, R_6$, $D_{1,2}, \ldots, D_{6,1}$, $F_{1,2}, \ldots, F_{6,1}$, W and Z. We call them the basic sets. We note that:

$$w \text{ is complete to } R_1 \cup R_2 \cup R_4 \cup R_5 \cup D_{2,3} \cup D_{3,4} \cup D_{5,6} \cup D_{6,1} \cup F_{1,2} \cup F_{4,5}. \quad (6)$$

Suppose that w has a non-neighbor u in that set. Up to symmetry, let $u \in R_1 \cup D_{3,4} \cup F_{4,5}$. Then u-r_3-r_1-w-r_2 is a P_5. Thus (6) holds.

It is convenient here to rename w as r_7. For any two integers i and j in $\{1, \ldots, 7\}$, let $N_{i,j}$ be the set of vertices that are complete to $\{r_i, r_j\}$. We observe that if r_i and r_j are adjacent, then N_{ij} is a stable set, for if it contained two adjacent vertices u and v, then $\{u, v, r_i, r_j\}$ would induce a K_4 in G. Thus we know that:

> $N_{1,3}$, $N_{1,5}$, $N_{3,5}$, $N_{2,4}$, $N_{2,6}$, $N_{4,6}$, $N_{1,4}$, $N_{2,5}$, $N_{3,6}$, $N_{1,7}$, $N_{2,7}$, $N_{4,7}$, $N_{5,7}$ are stable sets. (7)

Note that by (6), each of the basic sets, except for $D_{1,2}$ and $D_{4,5}$, is included in one of the sets in (7). So these eighteen sets are stable sets. In addition:

> $D_{4,5}$ and $D_{1,2}$ are stable sets. Moreover, $D_{4,5}$ is anticomplete to each of the nine sets R_1, R_2, R_3, R_6, W, $D_{3,4}$, $D_{5,6}$, $F_{3,4}$ and $F_{5,6}$. Likewise, $D_{1,2}$ is anticomplete to R_3, R_4, R_5, R_6, W, $D_{6,1}$, $D_{2,3}$, $F_{6,1}$ and $F_{2,3}$. (8)

If $D_{4,5}$ contains two adjacent vertices t and t', then $\{t, t', r_3, r_4, r_5, r_6\}$ induces a simple diamond. The same holds for $D_{1,2}$. Hence they are stable sets. Now suppose that there is an edge tx such that $t \in D_{4,5}$ and x lies in any of the nine sets in the second sentence of (8). In any of the nine cases, there is a P_3 r-r'-r'' on the 5-hole induced by $\{r_1, r_2, r_3, r_6, w\}$ such that x is adjacent to r and not to r' or r'' (if $x \in R_1 \cup D_{3,4}$, take w-r_2-r_6; if $x \in R_3 \cup F_{5,6}$, take r_1-w-r_2; if $x \in W$, take r_1-r_3-r_6; the other cases are symmetric). Then t-x-r-r'-r'' is a P_5. The proof is similar for $D_{1,2}$. Thus (8) holds.

> $D_{3,4}$ is complete to $D_{5,6}$, $D_{6,1}$ is complete to $D_{2,3}$, and $D_{3,4} \cup D_{5,6}$ is anticomplete to $D_{6,1} \cup D_{2,3}$. Moreover, one of $D_{3,4}$, $D_{5,6}$, $D_{6,1}$, $D_{2,3}$ is empty. (9)

Pick any vertex $d_{i,i+1}$ in $D_{i,i+1}$ for each i in $\{2, 3, 5, 6\}$. Then $d_{3,4}d_{5,6}$ is an edge, for otherwise $d_{3,4}$-r_4-r_1-r_5-$d_{5,6}$ is a P_5. So $D_{3,4}$ is complete to $D_{5,6}$. Likewise, $D_{6,1}$ is complete to $D_{2,3}$. Next, $d_{3,4}d_{6,1}$ is not an edge, for otherwise $d_{3,4}$-$d_{6,1}$-r_1-r_5-r_2 is a P_5, and $d_{3,4}d_{2,3}$ is not an edge, for otherwise $d_{3,4}$-$d_{2,3}$-r_2-r_5-r_1 is a P_5. Thus $D_{3,4}$ is anticomplete to $D_{6,1} \cup D_{2,3}$, and so is $D_{5,6}$, by symmetry.

Finally, if all four vertices $d_{3,4}, d_{5,6}, d_{6,1}, d_{2,3}$ do exist, then $d_{3,4}$-$d_{5,6}$-r_6-$d_{6,1}$-$d_{2,3}$ is a P_5. Thus (9) holds.

$$D_{2,3} \cup D_{3,4} \cup R_6 \text{ and } D_{5,6} \cup D_{6,1} \cup R_3 \text{ are stable sets.} \tag{10}$$

Suppose on the contrary that there are adjacent vertices u and v in $D_{2,3} \cup D_{3,4} \cup R_6$. Since each of the basic sets $D_{2,3}, D_{3,4}, R_6$ is a stable set, we may assume up to symmetry that $u \in D_{3,4}$ and $v \in R_6 \cup D_{2,3}$. Then u-v-r_2-r_5-r_1 is a P_5. The proof is similar for $D_{5,6} \cup D_{6,1} \cup R_3$. Thus (10) holds.

(There is a number of other notable facts, for example: $F_{3,4} \cup F_{5,6}$ is anticomplete to $F_{6,1} \cup F_{2,3}$; and one of $F_{3,4}$, $F_{5,6}$, $F_{6,1}$, $F_{2,3}$ is empty; but we will not use them.)

We now show that $G \setminus Z$ is 5-colorable. By (9), we may assume that $D_{2,3} = \emptyset$. Let $F^*_{5,6} = \{v \in F_{5,6} \mid v \text{ has a neighbor in } D_{5,6}\}$. Let:
- $S_1 = R_1 \cup R_2 \cup D_{3,4} \cup F_{4,5}$,
- $S_2 = R_4 \cup R_5 \cup D_{6,1} \cup F_{1,2}$,
- $S_3 = R_6 \cup F_{3,4} \cup F_{2,3}$,
- $S_4 = W \cup F_{6,1} \cup F^*_{5,6}$,
- $S_5 = R_3 \cup D_{5,6} \cup (F_{5,6} \setminus F^*_{5,6}) \cup D_{4,5}$.

So $S_1, \dots, S_5$ form a partition of $V(G) \setminus (Z \cup D_{1,2})$. The five sets $S_1, \dots, S_5$ are depicted in Figure 4, where edges of the complement of G are depicted instead of edges of G to make the picture more readable. We observe that, by the definition of the basic sets and by (6), we have $S_1 \subseteq N_{4,7}$, $S_2 \subseteq N_{1,7}$, $S_3 \subseteq N_{2,4}$, and $S_4 \subseteq N_{1,5}$, so, by (7), S_1, S_2, S_3 and S_4 are stable sets. Concerning S_5, we know that $R_3 \cup D_{5,6}$ is a stable set by (10), and $R_3 \cup F_{5,6}$ is a stable set as it is included in $N_{1,5}$; and the definition of $F^*_{5,6}$ and (8) imply that S_5 is a stable set. So $S_1, \dots, S_5$ form a 5-coloring of $G \setminus (Z \cup D_{1,2})$.

Now consider any vertex t in $D_{1,2}$. Suppose that t has a neighbor x in S_3 and a neighbor y in S_4. By (8), we have $x \in F_{3,4}$ and $y \in F^*_{5,6}$. By the definition of $F^*_{5,6}$, y has a neighbor u in $D_{5,6}$. Then xy is an edge, for otherwise x-r_2-w-r_1-y is a P_5, also xu is an edge, for otherwise x-r_3-r_1-w-u is a P_5, and tu is an edge, for otherwise t-r_1-r_3-r_6-u is a P_5. But then $\{t, x, y, u\}$ induces a K_4. So t is anticomplete to S_3 or to S_4, and t can receive the corresponding color. Thus we obtain a 5-coloring of $G \setminus Z$.

Now consider Z. By Lemma 3.2, $G[Z]$ is 3-colorable. Moreover:

$$N(Z) \subseteq F_{1,2} \cup F_{4,5}. \tag{11}$$

Consider any edge zt with $z \in Z$ and $t \notin Z$ and suppose that $t \notin F_{1,2} \cup F_{4,5}$. So t is in R_i or $D_{i,i+1}$ for some i in $\{1, \dots, 6\}$ or in $F_{j,j+1}$ for some j in $\{2, 3, 5, 6\}$. If $t \in R_1$, then z-t-r_3-r_6-r_2 is a P_5. If $t \in D_{1,2}$, then z-t-r_1-r_3-r_6 is a P_5. If $t \in F_{2,3}$, then z-t-r_3-r_5-w is a P_5. The other (symmetric) cases are similar. Thus (11) holds.

Recall that vertices of $F_{1,2} \cup F_{4,5}$ receive colors 1 and 2. By (11), Z may receive colors 3, 4 and 5. This completes the proof of the theorem. □

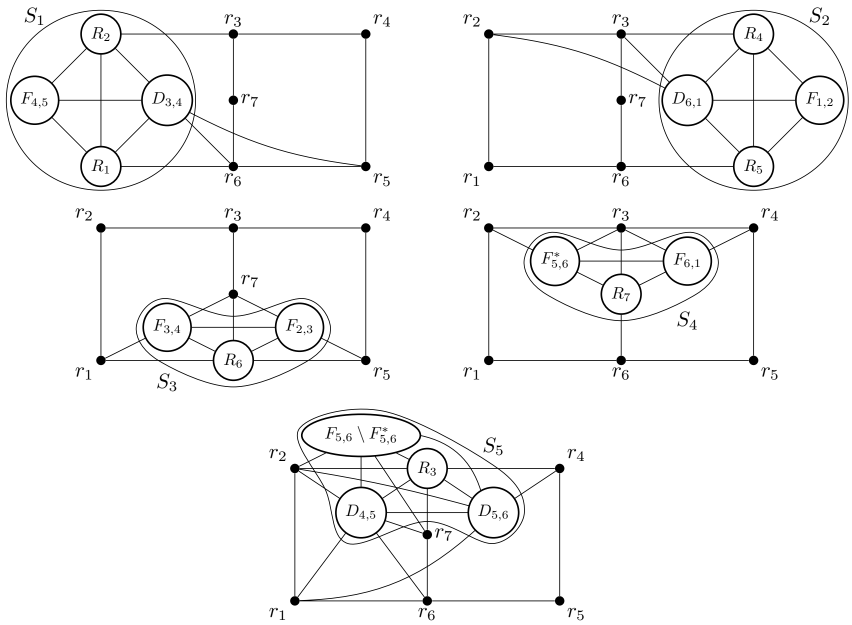

Figure 4: A 5-coloring of $G \setminus (Z \cup D_{1,2})$ when G contains a ruby. The picture shows the complement $\overline{G}$ of G. A line between two sets does not necessarily mean that they are complete to each other in $\overline{G}$. Also some adjacency may be unrepresented. Each circled set is a clique in $\overline{G}$.

6 5-holes

Now we can prove that every $\{P_5, K_4\}$-free graph that contains a 5-hole is 5-colorable.

A *solitaire* is a graph with six vertices $c_1, \dots, c_5, f$ such that $\{c_1, \dots, c_5\}$ induces a C_5 and f is adjacent to at least four vertices in that set. Let us say that the solitaire is *special* if f is adjacent to exactly four vertices of $\{c_1, \dots, c_5\}$.

Theorem 6.1. *Let G be a $\{P_5, K_4\}$-free that contains a C_5. Then G is* 5-*colorable.*

Proof. By Theorems 4.4, 5.2 and 5.4 we may assume that G contains no simple diamond, no sapphire and no ruby. Let $C = \{c_1, \dots, c_5\}$ be the vertex-set of a C_5 in G, with edges c_ic_{i+1} (modulo 5). Without loss of generality, we may assume that if G contains a solitaire, then there exists a vertex f such that $C \cup \{f\}$ itself induces a solitaire, and if G contains a special solitaire, then $C \cup \{f\}$ induces a special solitaire where f is not adjacent to c_5. In either case, we define sets as follows. For each $i \in \{1, \dots, 5\}$, let:

- $R_i = \{x \in V(G) \mid N_C(x) = \{c_{i-1}, c_{i+1}\}\}$;
- $Y_i = \{x \in V(G) \mid N_C(x) = \{c_{i-2}, c_i, c_{i+2}\}\}$;
- $F = \{x \in V(G) \mid \{c_1, c_2, c_3, c_4\} \subseteq N(x)\}$;
- $Z = \{x \in V(G) \mid N_C(x) = \emptyset\}$.

Clearly, the sets R_1, …, R_5, Y_1, …, Y_5, F and Z are pairwise disjoint. Note that G contains a solitaire if and only if $F \neq \emptyset$. We claim that:

$$V(G) = R_1 \cup \cdots \cup R_5 \cup Y_1 \cup \cdots \cup Y_5 \cup F \cup Z. \tag{12}$$

By Lemma 3.1, every vertex x of $V(G) \setminus C$ is of type 0, 2t, 3, 3y, 4 or 5 on C. If x is of type 0, then $x \in Z$. If x is of type 2t, then $x \in R_i$ for some i. If x is of type 3, then $C \cup \{x\}$ induces a simple diamond, a contradiction. If x is of type 3y, then $x \in Y_i$ for some i. If x is of type 4, with $N_C(x) = C \setminus \{c_j\}$ for some j, then $C \cup \{x\}$ induces a special solitaire, so f exists and is not adjacent to c_5. If $j \in \{1, 2\}$, then xf is not an edge, for otherwise $\{x, f, c_3, c_4\}$ induces a K_4; but then $C \cup \{f, x\}$ induces either a ruby (if $j = 1$) or a sapphire (if $j = 2$), a contradiction. So $j \notin \{1, 2\}$, and by symmetry $j \notin \{3, 4\}$. Thus we have $j = 5$ and $x \in F$. Finally, if x is of type 5 then $x \in F$. Thus (12) holds.

$$R_1, \ldots, R_5, Y_2, Y_3, F \cup Y_1, F \cup Y_4 \text{ and } F \cup Y_5 \text{ are stable sets.} \tag{13}$$

Suppose that there are two adjacent vertices u and v in one of these sets. If $u, v \in R_i$ for some i in $\{1, \ldots, 5\}$, then $\{u, v, c_{i+1}, c_{i+2}, c_{i-2}, c_{i-1}\}$ induces a simple diamond. If $u, v \in Y_i$ for some i in $\{1, \ldots, 5\}$, then $\{u, v, c_{i-2}, c_{i+2}\}$ induces a K_4. If $u, v \in F \cup Y_i$ with i in $\{1, 4, 5\}$, then $\{u, v, c_{i-2}, c_{i+2}\}$ induces a K_4. Thus (13) holds.

Now let us show that:

The subgraph $G \setminus Z$ admits a 5-coloring where each of the sets $Y_1, \ldots, Y_5$ and F receives only one color. (14)

In order to prove (14), we distinguish between three cases.

Case 1: f exists and is not adjacent to c_5. Let:
$B_3 = \{x \in R_3 \mid x \text{ has a neighbor in } R_1 \cup Y_2\}$ and $S_1 = R_1 \cup Y_2 \cup (R_3 \setminus B_3)$;
$B_2 = \{x \in R_2 \mid x \text{ has a neighbor in } R_4 \cup Y_3\}$ and $S_2 = R_4 \cup Y_3 \cup (R_2 \setminus B_2)$;
$S_3 = F \cup Y_5$, $S_4 = B_2 \cup Y_1$, and $S_5 = B_3 \cup Y_4$. We claim that:

Each of $S_1, \ldots, S_5$ is a stable set. Moreover, every vertex of R_5 is anticomplete to S_4 or to S_5. (15)

First suppose that, for some h in $\{1, \ldots, 5\}$, the set S_h contains two adjacent vertices u and v. First let $h = 1$. By (13) and the definition of B_3, we have $u \in R_1$ and $v \in Y_2$. Then fx is an edge for each x in $\{u, v\}$, for otherwise x-c_5-c_1-f-c_3 is a P_5. But then $\{f, u, v, c_2\}$ induces a K_4. The proof is similar (by symmetry) for $h = 2$. If $h = 3$, then $\{u, v, c_2, c_3\}$ induces a K_4. Now let $h = 4$. By (13) we have $u \in B_2$ and $v \in Y_1$, so u has a neighbor s in $R_4 \cup Y_3$, and fv is not an edge. Then fs is an edge, for otherwise s-c_5-c_4-f-c_2 is a P_5, and

fu is an edge, for otherwise u-v-c_4-f-c_2 is a P_5. But then $\{f, u, s, c_3\}$ induces a K_4. If $h = 5$ the proof is similar. This establishes the first sentence of (15).
Now suppose that some vertex x in R_5 has neighbors u and v with $u \in S_4$ and $v \in S_5$. If $u \in Y_1$ and $v \in Y_4$, then uv is not an edge, for otherwise $\{u, v, x, c_1\}$ induces a K_4; but then $\{x, u, v, c_1, c_2, c_3, c_4\}$ induces a ruby. Thus we may assume, up to symmetry, that $u \in B_2$, and so u has a neighbor s in $R_4 \cup Y_3$. Then fs is an edge, for otherwise s-c_5-c_4-f-c_2 is a P_5, and fu is not an edge, for otherwise $\{f, u, s, c_3\}$ induces a K_4. Also fx is an edge, for otherwise u-x-c_4-f-c_2 is a P_5, and vs is an edge, for otherwise s-c_5-c_4-v-c_2 is a P_5. Suppose that $v \in Y_4$ (so vc_1 is an edge). Then uv is not an edge, for otherwise $\{u, v, x, c_1\}$ induces a K_4. If s has no neighbor in $\{c_1, x\}$, then s-c_3-c_4-x-c_1 is a P_5. If s is adjacent to both c_1 and x, then $\{s, c_1, x, v\}$ induces a K_4. If s is adajcent to c_1 and not to x, then $\{x, u, v, s, c_1, c_3, c_4\}$ induces a ruby. If s is adjacent to x and not to c_1, then $\{x, u, v, s, c_1, c_2, c_3\}$ induces a ruby. Now suppose that $v \in B_3$, so vc_1 is an edge, and v has a neighbor t in $R_1 \cup Y_2$. This restores the symmetry, and so we know that ft and ut are edges and fv is not an edge. Then uv is an edge, for otherwise v-c_4-c_5-c_1-u is a P_5, and st is not an edge, for otherwise $\{u, v, s, t\}$ induces a K_4. Then xs is not an edge, for otherwise $\{x, u, v, s\}$ induces a K_4. Similarly, xt is not an edge. Then sc_1 is an edge, for otherwise x-c_1-c_2-c_3-s is a P_5. Similarly, tc_4 is an edge. But then s-c_1-c_2-t-c_4 is a P_5. Thus (15) holds.

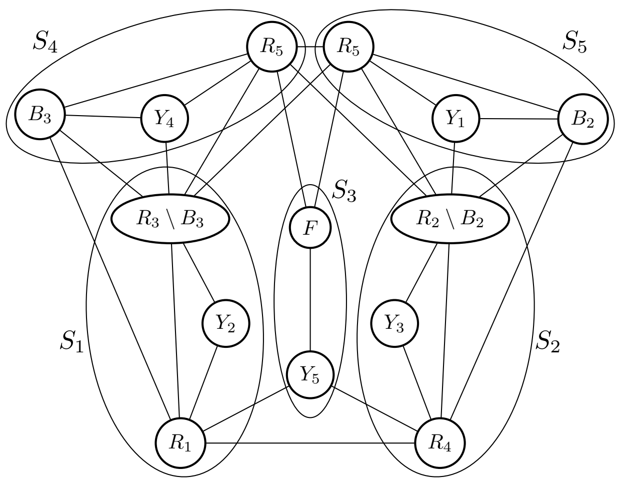

Figure 5: A 5-coloring of $G \backslash Z$ when G contains a special solitaire (case 1). The picture shows the complement $\overline{G}$ of G. A line between two sets does not necessarily mean that they are complete to each other in $\overline{G}$. Also some adjacency may be unrepresented. Each circled set is a clique in $\overline{G}$.

By (15), we can obtain a 5-coloring of $G \setminus Z$ starting from $S_1, \dots, S_5$ and adding each vertex of R_5 to S_4 or S_5 (see Figure 5, where the complement of G

is depicted instead of G).

Case 2: f exists and is adjacent to c_5. Hence G contains no special solitaire. We claim that:

For each i in $\{1, \dots, 5\}$, F is complete to R_i. Moreover, $R_i \cup R_{i+2}$ and $R_i \cup Y_{i-1} \cup Y_{i+1}$ are stable sets. (16)

First suppose that there are non-adjacent vertices v and r with $v \in F$ and $r \in R_i$. Since G contains no special solitaire, v is adjacent to c_5. Then $\{v, r, c_{i+1}, c_{i+2}, c_{i+3}, c_{i+4}\}$ induces a special solitaire. Now suppose that there are adjacent vertices u and v in one of the sets mentioned in the second sentence of (16). If $u, v \in R_i \cup R_{i+2}$, then $\{u, v, f, c_{i+1}\}$ induces a K_4. If $u, v \in R_i \cup Y_{i-1} \cup Y_{i+1}$, then, by (13) and up to symmetry, we have $u \in Y_{i-1}$ and $v \in R_i \cup Y_{i+1}$. If $v \in R_i$, then $\{u, v, c_{i+1}, c_{i+2}, c_{i+3}, c_{i+4}\}$ induces a special solitaire. If $v \in Y_{i+1}$, then fv is not an edge, for otherwise $\{v, f, c_{i-2}, c_{i-1}\}$ is a K_4; but then u-v-c_{i-2}-f-c_i induces a P_5. Thus (16) holds.

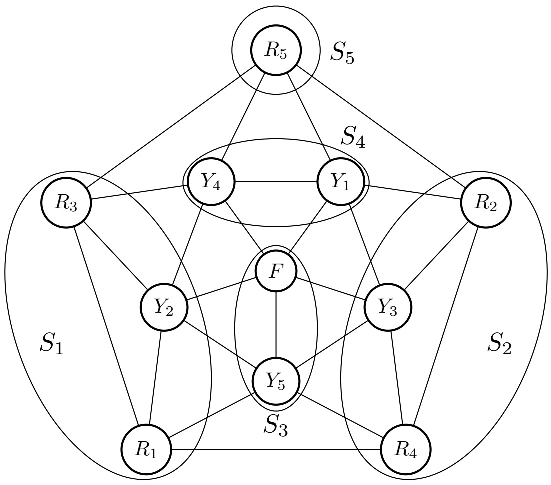

Figure 6: A 5-coloring of $G \setminus Z$ when G contains a solitaire (case 2). The picture shows the complement $\overline{G}$ of G. A line between two sets does not necessarily mean that they are complete to each other in $\overline{G}$. Also some adjacency may be unrepresented. Each circled set is a clique in $\overline{G}$.

Let $S_1 = R_1 \cup Y_2 \cup R_3$, $S_2 = R_4 \cup Y_3 \cup R_2$, $S_3 = F \cup Y_5$, $S_4 = Y_1 \cup Y_4$ and $S_5 = R_5$. It follows from (13) and (16) that S_1, S_2, S_3, S_4, S_5 form a 5-coloring of $G \setminus Z$ (see Figure 6, where the complement of G is depicted instead of G).

Case 3: $F = \emptyset$. Hence G contains no solitaire. For each i in $\{1, \dots, 5\}$, let $S_i = R_i \cup Y_{i+1}$. If S_i contains two adjacent vertices u and v, then, by (13), we have $u \in R_i$ and $v \in Y_{i+1}$, and $\{u, v, c_{i+1}, c_{i+2}, c_{i+3}, c_{i+4}\}$ induces a solitaire, a contradiction. So S_i is a stable set. It follows that $S_1, \ldots, S_5$ form a 5-coloring of $G \setminus Z$. This completes the proof of (14).

All that remains is to extend the 5-coloring of $G \setminus Z$ we obtained in each of the three cases to Z. We have:

$$N(Z) \subseteq Y_1 \cup \cdots \cup Y_5 \cup F. \tag{17}$$

Suppose that there is an edge zt with $z \in Z$ and $t \notin Z \cup Y_1 \cup \cdots \cup Y_5 \cup F$. So we have $t \in R_i$ for some i in $\{1, \ldots, 5\}$, and then z-t-c_{i+1}-c_{i+2}-c_{i+3} is a P_5, a contradiction. Thus (17) holds.

We now extend the 5-coloring of $G \setminus Z$ to each component X of Z, as follows.

First assume that X has only one vertex x. If x has no neighbor of some color, then this color can be assigned to x. So suppose that x has a neighbor of color i for each i in $\{1, \ldots, 5\}$. This implies that we are in Case 1 or 3, because in Case 2 one color is used only in R_5, and by (17) we know that x has no neighbor in R_5. Suppose that we are in Case 3. By (17), x has a neighbor y_i in Y_i for each i. For each i, y_iy_{i+1} must be an edge, for otherwise y_i-c_{i+2}-c_{i+1}-y_{i+1}-c_{i-1} is a P_5, and y_iy_{i+2} is not an edge, for otherwise $\{x, y_i, y_{i+1}, y_{i+2}\}$ induces a K_4. But then $\{x, y_1, \ldots, y_5\}$ induces a solitaire, a contradiction. Now suppose that we are in Case 1. So x has a neighbor y_i in Y_i for each i in $\{1, \ldots, 4\}$ (and x has a neighbor in $F \cup Y_5$). We see that fy_2 is an edge, for otherwise y_2-c_5-c_1-f-c_3 is a P_5. Similarly, fy_3 is an edge. Also fx is an edge, for otherwise c_5-y_3-f-c_2-x is a P_5, and y_2y_3 is an edge, for otherwise y_2-c_4-c_3-y_3-c_1 is a P_5. But then $\{f, x, y_2, y_3\}$ induces a K_4.

Now assume that $|X| \geq 2$.

Every vertex of $Y_1 \cup \cdots \cup Y_5$ is either complete or anticomplete to X. Moreover, X is adjacent to at most one of $Y_1, \ldots, Y_5$. (18)

Note that we cannot apply Lemma 3.2 directly to obtain the first part of the claim, since C does not contain any triangle. If, for any i in $\{1, \ldots, 5\}$, a vertex y in Y_i is neither complete nor anticomplete to X, then there are adjacent vertices u and v in X such that y is adjacent to u and not to v, but then v-u-y-c_i-c_{i+1} is a P_5. Thus the first sentence of (18) holds. Now suppose that X has neighbors y_i and y_j with $y_i \in Y_i$, $y_j \in Y_j$, $i, j \in \{1, \ldots, 5\}$ and $i \neq j$. Let u, v be two adjacent vertices in X. By the preceding point, y_i and y_j are complete to $\{u, v\}$. So y_iy_j is not an edge, for otherwise $\{u, v, y_i, y_j\}$ induces a K_4. Up to symmetry, let $j \in \{i+1, i+2\}$. If $j = i+1$, then y_j-c_{i-1}-c_i-y_i-y_{i+2} is a P_5. If $j = i+2$, then $\{u, v, y_i, y_j, c_{i-2}, c_{i-1}\}$ induces a simple diamond. Thus (18) holds.

By (17) and (18), we have $N(X) \subseteq Y_i \cup F$ for some i in $\{1, \ldots, 5\}$. By (14) and up to relabelling, we may assume that every vertex of $Y_i \cup F$ has color 3 or 5. Let $X' = \{x \in X \mid x \text{ has a neighbor in } Y_i \cup F\}$ and $X'' = X \setminus X'$. Let a be a vertex of $Y_i \cup F$ with the largest number of neighbors in X'. We claim that a is complete to X'. For suppose that a has a non-neighbor x_a in X'. By

the definition of X', x_a has a neighbor b in $Y_i \cup F$. By the choice of a, there is a vertex x_b in X' that is adjacent to a and not to b. By (18), a and b are in F, and by (13) they are not adjacent. If x_a and x_b are not adjacent, then x_a-b-c_1-a-x_b is a P_5, while if they are adjacent then $\{x_a, x_b, a, b, c_1, c_2\}$ induces a simple diamond. This proves the claim that a is complete to X'. Therefore $G[X']$ contains no K_3, and by Theorem 2.1 it is 3-colorable. We color $G[X']$ with colors $1, 2$ and 4, using color 4 only on those components of $G[X']$ that are 5-rings.
Finally, consider any component W of X''. By Lemma 3.2 (applied to $C \cup Y_i \cup F$ and X''), W is homogeneous. Since $G[X]$ is connected, there is a vertex t in X' adjacent to W. Since W is homogeneous, t is complete to W, so $G[W]$ contains no K_3, and by Theorem 2.1 it is 3-colorable. We color W with colors $3, 4$ and 5, using color 3 or 5 if W has only one vertex. If this is not a proper coloring, then it can only be because color 4 was assigned to two adjacent vertices x and w with $x \in X'$ and $w \in X''$. By the definition of the coloring x belongs to a component of $G[X']$ that is a 5-ring, so x lies on a 5-hole C_x in $G[X']$; and w is in a component W of X'' of size at least 2, so w has a neighbor w'. If w is adjacent to two consecutive vertices u and v of C_x, then, since W is homogeneous, $\{u, v, w, w'\}$ induces a K_4. In the opposite case, by Lemma 3.1, w must be of type 2t on C. But then $C \cup \{w, w'\}$ contains a simple diamond, a contradiction. Thus we have a proper 5-coloring of G. This completes the proof of the theorem. □

7 Antiholes

Here is a proof of Theorem 1.1 that does not use Theorem 1.2. Recall that a graph G is *perfect* if every induced subgraph G' of G satisfies $\chi(G') = \omega(G')$. Graphs with no k-hole and no k-antihole for any $k \geq 5$ are called *weakly chordal*. Hayward [10] proved that every weakly chordal graph is perfect. Now let G be any $\{P_5, K_4\}$-free graph. We know that G contains no k-hole with $k \geq 6$ and no ℓ-antihole with $\ell \geq 8$. If G is weakly chordal, then G is 3-colorable by Hayward's theorem. If G is not weakly chordal, it must contain either a 5-hole, a 7-antihole of a 6-antihole, and the result follows from our Theorems 6.1, 7.1 and 7.2.

Theorem 7.1. *Let G be a $\{P_5, K_4\}$-free that contains a 7-antihole. Then G is 5-colorable.*

Proof. By Theorem 6.1, we may assume that G contains no 5-hole. Let $C = \{c_1, \ldots, c_7\}$ be the vertex-set of a 7-antihole in G, with non-edges $c_i c_{i+1}$ (modulo 7). For each integer i in $\{1, \ldots, 7\}$, let $R_i = \{x \in V(G) \mid N_C(x) = N_C(c_i)\}\}$ and $T_i = \{x \in V(G) \mid N_C(x) = \{c_{i-1}, c_i, c_{i+1}\}\}$. We claim that:

$$V(G) = R_1 \cup \cdots \cup R_7 \cup T_1 \cup \cdots \cup T_7. \tag{19}$$

Clearly, $c_i \in R_i$ for each i. Now consider any vertex x in $V(G) \setminus C$. Let $n = |N_C(x)|$. Suppose that $n \geq 5$. So there are two integers i, j such that x

is complete to $C \setminus \{c_i, c_j\}$, and we may assume that $j \in \{i+1, i+2, i+3\}$. If $j = i+1$ or $i+3$, then $\{x, c_{i+2}, c_{i+4}, c_{i+6}\}$ induces a K_4; if $j = i+2$, then $\{x, c_{i+1}, c_{i+4}, c_{i+6}\}$ induces a K_4. So we must have $n \leq 4$.
Suppose that $n = 4$. Let c_i, c_j, c_k ($i, j, k \in \{1, \ldots, 7\}$) be the three non-neighbors of x in C. Suppose that i, j, k are consecutive integers (modulo 7), say $k = j+1 = i+2$. Then $N_C(x) = N_C(c_j)$, so $x \in R_j$. Now suppose that i, j are consecutive integers but i, j, k are not; so, up to symmetry, $j = i+1$ and $k \in \{j+2, j+3\}$. If $k = j+2$, then x-c_{j+1}-c_i-c_k-c_j is a P_5. If $k = j+3$, then $\{x, c_k, c_{k+1}, c_{k+2}, c_{k+3}\}$ induces a 5-hole. Finally, suppose that no two of i, j, k are consecutive; so, up to symmetry, we have $k = j+2 = i+4$. Then $\{x, c_{i+1}, c_{i+3}, c_{i+5}\}$ induces a K_4.
Suppose that $n = 3$. Let c_i, c_j, c_k be the three neighbors of x in C. If i, j, k are consecutive integers, say $k = j+1 = i+2$, then $x \in T_j$. If i, j are consecutive integers but i, j, k are not, then, up to symmetry, we have $j = i+1$ and $k \in \{j+2, j+3\}$, and then x-c_k-c_{k+2}-c_{k-1}-c_{k+1} is a P_5. Finally, if no two of i, j, k are consecutive, then $\{x, c_i, c_j, c_k\}$ induces a K_4.
Suppose that $n = 2$. Let c_i, c_j be the two neighbors of x in C, with (up to symmetry) $j \in \{i+1, i+2, i+3\}$. If $j = i+1$, then $\{x, c_{i-1}, c_i, c_{i+1}, c_{i+2}\}$ induces a 5-hole. If $j \in \{i+2, i+3\}$, then x-c_j-c_{j+2}-c_{j-1}-c_{j+1} is a P_5.
Suppose that $n = 1$. Let c_i be the neighbor of x in C. Then x-c_i-c_{i+2}-c_{i-1}-c_{i+1} is a P_5.
Suppose that $n = 0$. So x belongs to the set Z of vertices that have no neighbor in C. Since G is connected, there is an edge zt with $z \in Z$ and $t \notin Z$. By the preceding arguments, we have $t \in R_i \cup T_{i+3}$ for some i. Then z-t-c_{i+2}-c_{i-1}-c_{i+1} is a P_5. Thus Z is empty and (19) holds.

For each $i \in \{1, \ldots, 7\}$, $R_{i-3} \cup R_{i+3} \cup T_i$ and $T_i \cup T_{i+1}$ are stable sets. (20)

By the definition of $R_{i-3} \cup R_{i+3} \cup T_i$, this set is complete to $\{c_{i-1}, c_{i+1}\}$; so if it contains two adjacent vertices x and x', then $\{x, x', c_{i-1}, c_{i+1}\}$ induces a K_4. Secondly, If there exist adjacent vertices t and t' in $T_i \cup T_{i+1}$, then, by the preceding sentence, we have $t \in T_i$ and $t' \in T_{i+1}$ and then t-t'-c_{i+2}-c_{i-2}-c_{i+3} is a P_5. Thus (20) holds.

It follows from (19) and (20) that the five sets $R_7 \cup T_3 \cup T_4$, $R_1 \cup R_2 \cup T_5$, $R_3 \cup T_6 \cup T_7$, $R_4 \cup R_5 \cup T_1$ and $R_6 \cup T_2$ are stable sets, and so they form a 5-coloring of G. □

Theorem 7.2. *Let G be a $\{P_5, K_4\}$-free that contains a 6-antihole. Then G is 5-colorable.*

Proof. By Theorems 7.1 and 6.1, we may assume that G contains no 5-hole and no 7-antihole. Let $C = \{c_1, \ldots, c_6\}$ be the vertex-set of a 6-antihole in G, with non-edges $c_i c_{i+1}$ (modulo 6). For each integer i in $\{1, \ldots, 6\}$, let:
- $R_i = \{x \in V(G) \mid N_C(x) = N_C(c_i)\}$,
- $F_{i,i+1} = \{x \in V(G) \mid N_C(x) = \{c_{i+2}, c_{i+3}, c_{i+4}, c_{i+5}\}\}$,
- $Z = \{x \in V(G) \mid N_C(x) = \emptyset\}$.

We claim that:

$$V(G) = R_1 \cup \cdots \cup R_6 \cup F_{1,2} \cup \cdots \cup F_{6,1} \cup Z. \tag{21}$$

Clearly, $r_i \in R_i$ for each i. Now consider any vertex x in $V(G) \setminus C$. Let $A = \{c_1, c_3, c_5\}$, $B = \{c_2, c_4, c_6\}$, $n_A = |N_A(x)|$ and $n_B = |N_B(x)|$. If $n_A = 3$, then $A \cup \{x\}$ induces a K_4. So $n_A \leq 2$, and similarly $n_B \leq 2$. If $n_A = 0$ and $n_B = 0$, then $x \in Z$. Now suppose that $n_A = 2$. Up to symmetry, let $N_A(x) = \{c_1, c_3\}$. Then xc_2 is an edge, for otherwise $\{x, c_1, c_5, c_2, c_6\}$ induces either a P_5 or a 5-hole; and x has at most one neighbor in $\{c_4, c_6\}$ since $n_B \leq 2$. Then, x is in R_5 or $F_{4,5}$ or $F_{5,6}$. The other (symmetric) cases are similar. Finally suppose that $n_A = 1$. By symmetry, we may assume that $n_B \leq 1$ and $N_A(x) = c_1$. Then either $\{x, c_1, c_3, c_6, c_2\}$ or $\{x, c_1, c_5, c_2, c_6\}$ induces a P_5 or a 5-hole. Thus (21) holds.

$$\text{For each } i \text{ in } \{1, \ldots, 6\},\ R_i \cup R_{i+1} \cup F_{i,i+1} \text{ is a stable set.} \tag{22}$$

If this set contains two adjacent vertices u and v, then $\{u, v, r_{i+2}, r_{i-1}\}$ induces a K_4. Thus (22) holds.

$$\text{For each } i \text{ in } \{1, \ldots, 6\}, \text{ either } F_{i-1,i} \text{ or } F_{i,i+1} \text{ is empty.} \tag{23}$$

Up to symmetry, let $i = 1$ and suppose that there are vertices u and v with $u \in F_{6,1}$ and $v \in F_{1,2}$. If uv is an edge, then $\{u, v, c_3, c_5\}$ induces a K_4. If it is not an edge, then $\{u, v\} \cup (C \setminus \{c_1\})$ induces a 7-antihole. Thus (23) holds.

By (23) and up to symmetry, we may assume that either (a) $F_{1,2} \cup F_{3,4} \cup F_{5,6} = \emptyset$ or (b) $F_{1,2} \cup F_{2,3} \cup F_{4,5} \cup F_{5,6} = \emptyset$.

By Lemma 3.1, we know that every component X of Z is homogeneous and 3-colorable. Moreover:

$$\text{For each component } X \text{ of } Z, \text{ there are two integers } i, j \text{ in } \{1, \ldots, 6\} \text{ such that } N(X) \subseteq F_{i,i+1} \cup F_{j,j+1}. \tag{24}$$

Suppose on the contrary that X has neighbors a, b and c in three sets $F_{i,i+1}$, $F_{j,j+1}$ and $F_{k,k+1}$, respectively, for three different values i, j, k in $\{1, \ldots, 6\}$. Then we must be in case (a), so $i = 2$, $j = 4$ and $k = 6$. We note that ab is an edge, for otherwise a-c_4-c_2-b-c_3 is a P_5, and similarly ac and bc are edges. Then for any vertex x in X, and since X is homogeneous, $\{x, a, b, c\}$ induces a K_4. Thus (24) holds.

Now let us show that G is 5-colorable. In case (a), the three sets $R_2 \cup R_3 \cup F_{2,3}$, $R_4 \cup R_5 \cup F_{4,5}$ and $R_6 \cup R_1 \cup F_{6,1}$ are stable sets by (23), so they form a 5-coloring of $G \setminus Z$. In case (b), the four sets $R_3 \cup R_4 \cup F_{3,4}$, $R_6 \cup R_1 \cup F_{6,1}$, R_2 and R_5 are stable sets by (23), so they form a 5-coloring of $G \setminus Z$. In either case, by (24) each component X of Z can be colored with three colors that are not present in $N(X)$. This completes the proof of the theorem. □

8 Conclusion

When G is any P_5-free graph, the proof from [8] plus the new fact, established here, that $f_{P_5}(3) = 5$, implies that $\chi(G) \leq 5 \cdot 3^{\omega(G)-3}$. Thus $f_{P_5}(\omega) \leq 3^{\omega - c}$, where $c = 3 - \frac{\log 5}{\log 3} \sim 1.535$.

Stéphan Thomassé asked the following: is it true that there exists a finite graph H with no K_4 and no P_5, such that any graph with no K_4 and no P_5 has a homomorphism to H? Most of the cases in the proof of our result suggest that this could be true, except the case of the simple diamond, where we do not end up with a nice homomorphism. This yields the following more general question. Given a hereditary class $\mathcal{C}$ with bounded chromatic number and closed under disjoint union, what conditions force the existence of a graph $H \in \mathcal{C}$, such that every graph of $\mathcal{C}$ has a homomorphism to H?

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