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Fast recognition of doubled graphs

Frédéric Maffray

Laboratoire G-SCOP
46 av. Félix Viallet, 38000 GRENOBLE, France
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Abstract

Double split graphs form one of the five basic classes in the proof of the strong perfect graph theorem by Chudnovsky, Robertson, Seymour and Thomas [The strong perfect graph theorem, *Annals of Mathematics* 164 (2006) 51–229]. A doubled graph is any induced subgraph of a double split graph. Alexeev, Fradkin and Kim [Forbidden induced subgraphs of double-split graphs, *SIAM J. Discrete Math.* 26 (2012) 1–14] established that the class of doubled graphs is characterized by a list of 44 forbidden induced graphs of order at most 9. It follows that deciding if a graph G is a doubled graph can be done in time $O(|V(G)|^9)$. We show here that this can be done in time $O(|V(G)|+|E(G)|)$ by analyzing the degree structure of the graph.

1 Introduction

The main result in the proof of the strong perfect graph theorem by Chudnovsky, Robertson, Seymour and Thomas [2] is a decomposition theorem for all Berge graphs. This theorem states that every Berge graph either admits one of several decompositions or belongs to one of five basic classes. The five basic classes are bipartite graphs, complements of bipartite graphs, line-graphs of bipartite graphs, complements of line-graphs of bipartite graphs, and double-split graphs. There are well-known characterizations by forbidden subgraphs and linear-time recognition algorithms for the first four classes [5, 6, 7]. We will focus here on the class of double-split graphs. Before giving the precise definition of this class, we need some terminology and notation.

All graphs considered here are finite and have no loops or multiple edges. Given a graph G, its vertex-set is denoted by $V(G)$ and its edge-set by $E(G)$. The complement of G is denoted by $\overline{G}$. The neighborhood of a vertex v (the set of vertices adjacent to v) is denoted by $N(x)$. The degree of v (the size of its neighborhood) is denoted by $d(v)$. We let P_n and C_n denote the path and cycle on n vertices, respectively. For any $S \subseteq V(G)$, we let $G[S]$ denote the subgraph of G induced by S. A *clique* is a set of pairwise adjacent vertices. A *stable set* is a set of pairwise non-adjacent vertices. A vertex v is *complete* to a set $S \subset V(G)$ if v is adjacent to every vertex in S, and *anticomplete* to S if it has no neighbor in S. Given a set $\mathcal{F}$ of graphs, a graph G is *$\mathcal{F}$-free* if no induced subgraph of G is isomorphic to any member of $\mathcal{F}$.

*CNRS, Laboratoire G-SCOP, INP, UJF, Grenoble, France (frederic.maffray@inpg.fr).

Following [1], we say that a graph G is *semi-matched* if every vertex in G has at most one neighbor (in other words, every component of G has size at most 2). A graph G is *semi-antimatched* if $\overline{G}$ is semi-matched. In a graph G we say that a set $S \subseteq V(G)$ is semi-matched (resp. semi-antimatched) if $G[S]$ is semi-matched (resp. semi-antimatched). Let $X, Y \subseteq V(G)$ be such that $X \cap Y = \emptyset$ and $X \cup Y = V(G)$. We say that the partition (X, Y) is *acceptable* if X is semi-matched, Y is semi-antimatched, and:
(a1) For all adjacent $u, v \in X$, every $y \in Y$ is adjacent to exactly one of u, v;
(a2) For all non-adjacent $v, w \in Y$, every $x \in X$ is adjacent to exactly one of v, w.

A graph is *split* if its vertex-set can be partitioned into a stable set and a clique. A graph is *double-split* if it admits an acceptable partition (X, Y) such that all components of $G[X]$ and of $\overline{G}[Y]$ have size 2 and both $G[X]$ and $\overline{G}[Y]$ have at least two components. A *doubled* graph is any induced subgraph of a double-split graph; in other words, any graph that admits an acceptable partition (X, Y). Note that a graph G is a doubled graph with acceptable partition (X, Y) if and only if $\overline{G}$ is a doubled graph with acceptable partition (Y, X). The class of double-split graphs is not closed under taking induced subgraphs, but the class of doubled graphs is, so it can be characterized by its family $\mathcal{F}$ of minimal forbidden induced subgraphs. Alexeev, Fradkin and Kim [1] determined the family $\mathcal{F}$; it consists of 44 graphs of order at most 9. We will not describe it here.

Theorem 1.1 ([1]) *A graph is a doubled graph if and only if it is $\mathcal{F}$-free.*

It follows from Theorem 1.1 that one can decide if a graph G is a doubled graph by testing every induced subgraph of G on at most 9 vertices; this takes time $O(|V(G)|^9)$. The question of the existence of a faster recognition algorithm was not addressed in [1], but it is raised in [8]. We will show that one can decide if a graph G is a doubled graph in time $O(|V(G)| + |E(G)|)$ (linear time) by analyzing the degree sequence of G.

We first observe that, given a partition (X, Y), we can determine if it is acceptable in linear time, as follows. Determine if X is semi-matched by building the graph $G[X]$ and checking that every vertex has degree at most 1 in $G[X]$. Likewise, determine if Y is semi-antimatched by building the graph $G[Y]$ and checking that every vertex has degree at least $|Y|-2$ in $G[Y]$. Now assume that X is semi-matched and Y is semi-antimatched. Determine if (X, Y) satisfies axiom (a1) in the definition of an acceptable partition as follows. If X is a stable set, there is nothing to do. If X is not a stable set, let $X_1, \ldots, X_h$ be the components of size 2 of $G[X]$. Initialize a counter $(c_1, \ldots, c_h) = (0, \ldots, 0)$. Pick a vertex $y \in Y$ and scan its adjacency list. For every neighbor z of y, if $z \in X_i$ for some $i \in \{1, \ldots, h\}$ then set $c_i := c_i + 1$. If $c_i = 2$, declare that the partition is not acceptable (indeed y has two neighbors in X_i) and stop. When the whole adjacency list of y is scanned and the algorithm has not stopped, check whether there is any j with $c_j = 0$; in that case, declare that the partition is not acceptable (indeed y has no neighbor in X_j) and stop. Else, reinitialize the counter. This procedure takes time $O(d(y))$ (in particular when we reinitialize the counter we have $h \leq d(y)$ since y has a neighbor in each X_i). Continue with every vertex in Y. In total this takes time $O(\Sigma_{y \in Y} d(y))$, which is linear time. Axiom (a2) can be checked similarly.

By the preceding argument, in order to decide if a graph is a doubled graph it suffices to guess a partition (X, Y) and to check whether it is acceptable or not. However, we might have to check $O(2^{|V(G)|})$ partitions. Our main result, Theorem 2.2 below, shows that there is only a constant number of partitions to check, whatever the input graph is. This will ensure that we can decide if a graph is a doubled graph in linear time.

2 The degree sequence

Let G be a graph on n vertices $v_1, \dots, v_n$ such that $d_1 \leq d_2 \leq \dots \leq d_n$, where $d_i = d(v_i)$ for all i. Let t be the largest integer such that $d_t \leq n - t$ (note that t exists because $d_1 \leq n - 1$). Define sets:

$$\begin{aligned} T &= \{v_1, \dots, v_t\}, \\ D &= \{v \mid d(v) \leq d_t\}, \\ D' &= \{v \mid d(v) < d_t\}, \\ D'' &= \{v \mid d(v) \leq d_t + 1\}. \end{aligned}$$

Recall that the degree sequence of a graph can be obtained in linear time, for example with bucket sort [3], so integers t and d_t and sets T, D, D' and D'' can be computed in linear time. Hammer and Simeone [4] observed that split graphs can be recognized in linear time on the basis of the following result.

Theorem 2.1 ([4]) *Let G be a graph on n vertices $v_1, \dots, v_n$ such that $d_1 \leq d_2 \leq \dots \leq d_n$, where $d_i = d(v_i)$ for all i. Let t and T be defined as above. Then G is a split graph if and only if T is a stable set and $V(G) \setminus T$ is a clique.*

Our main theorem is a generalization of Theorem 2.1. For this purpose, we need to introduce some more definitions. In a graph G, a 2-*vertex* is a vertex of degree 2 whose neighbors are not adjacent. A 3-*vertex* is a vertex of degree 3 whose neighborhood contains exactly one edge. Let integers t and d_t be defined as above. A d_t-*edge* is an edge whose vertices have degree 1 and d_t respectively. A d_t-*pair* is a pair of non-adjacent vertices of degree $n-2$ and d_t respectively. As we know the degree sequence, the 2-vertices, 3-vertices, d_t-edges and d_t-pairs of a graph can be found in linear time.

Let us say that a set $X \subseteq V(G)$ is *acceptable* if the partition $(X, V(G) \setminus X)$ is acceptable.

Theorem 2.2 *Let G be a graph on n vertices $v_1, \dots, v_n$ such that $d_1 \leq d_2 \leq \dots \leq d_n$, where $d_i = d(v_i)$ for all i. Let integers t and d_t and sets T, D, D' and D'' be defined as above. Suppose that G is not special. Then G is a doubled graph if and only if any of the candidate sets defined below is acceptable:*

1. *Sets $\emptyset$, $V(G)$, T, D, D' and D'' are candidates.*
2. *If $|D| \leq 4$, then every set $X \subseteq D$ with $|X| = 2$ is a candidate.*
3. *If $|D'| \geq n - 4$, then every set $X \supseteq D'$ with $|X| = n - 2$ is a candidate.*

4. *If there is exactly one component of size at least 2 in $G[D]$, and this component induces a path P_h on $h \in \{2,3,4\}$ vertices, with endvertices a and b, then $D \setminus \{a,b\}$ is a candidate; moreover if $h \in \{2,3\}$, then also $D \setminus \{a\}$ and $D \setminus \{b\}$ are candidates.*

5. *If G has exactly one 2-vertex x, with neighbors a, b, then $D' \cup \{x,a\}$, $D' \cup \{x,b\}$, $(D \setminus \{a\}) \cup \{b\}$ and $(D \setminus \{b\}) \cup \{a\}$ are candidates; moreover, if any $u \in \{a,b\}$ has exactly one neighbor v of degree d_t, then also $(D \setminus \{v\}) \cup \{u\}$ is a candidate.*

6. *If G has exactly two 2-vertices x and w, and there exists an induced P_5 a-x-b-w-c, then $D' \cup \{a,b\}$ is a candidate.*

7. *If G has exactly five 2-vertices, and they induce a P_5 v_1-v_2-v_3-v_4-v_5, then $D \setminus \{v_3\}$ is a candidate.*

8. *If G has exactly six 2-vertices $v_1, \dots, v_6$, and they induce a C_6 with edges $v_i v_{i+1}$ (mod 6), then $\{v_1, v_2, v_4, v_5\}$ is a candidate.*

9. *If G has at most six 3-vertices, then $D' \cup \{x, x'\}$ is a candidate for every 3-vertex x, where x' is the neighbor of x that has no neighbor in $N(x)$.*

10. *If G has exactly one or two d_t-edges, then $D' \cup A$ is a candidate, where A is the set of vertices of degree d_t in the d_t-edges.*

11. *If G has exactly one d_t-pair $\{u,v\}$, then $D \setminus \{v\}$ is a candidate (where $d(v) = d_t$).*

Proof. Clearly, if any candidate S is acceptable, then G is a doubled graph, with partition $(S, V(G) \setminus S)$. Conversely, assume that G is a doubled graph and let (X,Y) be a doubled partition of G. If X or Y is empty, then we are in item 1 (with candidate $\emptyset$ or $V(G)$). If G is a split graph, then T is acceptable by Theorem 2.1, so we are in item 1 (with candidate T). Therefore let us assume that G is not a split graph and that $X \neq \emptyset$ and $Y \neq \emptyset$. For each $h \in \{1,2\}$, let k_h be the number of components of $G[X]$ of size h, and let X_h be the union of their vertex-sets. Hence $X = X_1 \cup X_2$. Likewise, let ℓ_h be the number of components of $\overline{G}[Y]$ of size h, and let Y_h be the union of their vertex-sets. Hence $Y = Y_1 \cup Y_2$. The definition of a doubled graph implies easily that:

$$\ell_2 \le d(x) \le \ell_2 + \ell_1 \qquad (\forall x \in X_1) \tag{1}$$
$$1 + \ell_2 \le d(x) \le 1 + \ell_2 + \ell_1 \qquad (\forall x \in X_2) \tag{2}$$
$$2\ell_2 + \ell_1 - 1 + k_2 \le d(y) \le 2\ell_2 + \ell_1 - 1 + k_2 + k_1 \qquad (\forall y \in Y_1) \tag{3}$$
$$2\ell_2 + \ell_1 - 2 + k_2 \le d(y) \le 2\ell_2 + \ell_1 - 2 + k_2 + k_1 \qquad (\forall y \in Y_2). \tag{4}$$

Let $m_X = \max\{d(x), x \in X\}$ and let x^* be a vertex of X such that $d(x^*) = m_X$. Let $m_Y = \min\{d(y), y \in Y\}$ and let y^* be a vertex of Y such that $d(y^*) = m_Y$. By (1)–(4), we have $m_X \le \ell_1 + \ell_2 + 1$ and $m_Y \ge 2\ell_2 + \ell_1 - 2 + k_2$.

Suppose that $m_X < m_Y$. Let $k = |X|$. So $d(x^*) = d_k$ and $d(y^*) = d_{k+1}$. If $d_k \le n - k$ and $d_{k+1} > n - (k+1)$, then we have $t = k$ and $X = D$, so we are in item 1 (with candidate D). Now suppose that $d_k > n - k$. So $d(x^*) > |Y|$. This is possible only if x^* is complete to Y and lies in X_2, and consequently Y

is a clique (i.e., $Y_2 = \emptyset$, $\ell_2 = 0$); and so $d_k = \ell_1 + 1 = n - k + 1$. It follows that $t = k - 1$. If $d_{k-1} = \ell_1 + 1$, then $X = D$ and we are in item 1 again. If $d_{k-1} \leq \ell_1$, then $X = D''$ as x^* is the only vertex of degree $\ell_1 + 1$, so we are in item 1 (with candidate D''). Now suppose that $d_{k+1} \leq n - k - 1$. So $d(y^*) \leq |Y| - 1$. By (3)–(4), we know that $d(y^*) \geq |Y| - 2$. If $d(y^*) = |Y| - 1$, then $t = k + 1$ and $X = D'$, so we are in item 1. If $d(y^*) = |Y| - 2$, then either $t = k + 1$ (if $d_{k+2} > |Y| - 2$) or $t = k + 2$ (if $d_{k+2} = |Y| - 2$), and in either case $X = D'$ again.

We may now assume that $m_X \geq m_Y$. This implies $k_2 + \ell_2 \leq 3$. Let us examine this inequality in detail.

Case 1: $k_2 = 0$. Then $X_2 = \emptyset$, so $X = X_1$ and $m_X \leq \ell_1 + \ell_2$ by (1). The inequality $m_X \geq m_Y$ implies $\ell_2 \leq 2$. If $\ell_2 = 0$, then G is a split graph, so let us assume that $\ell_2 > 0$.

Suppose that $\ell_2 = 1$. So $n = k_1 + \ell_1 + 2$ and $m_X \leq \ell_1 + 1$. Let $Y_2 = \{y, y'\}$. We may assume that x^* is adjacent to y and not to y'. If y' has no neighbor in X, then G is a split graph (as $X \cup \{y'\}$ is a stable set and $Y \setminus \{y'\}$ is a clique). So y' has a neighbor in $X \setminus \{x^*\}$, and $k_1 \geq 2$. It follows that $m_Y \geq \ell_1 + 1$. Thus $m_X = m_Y = \ell_1 + 1$. Since x^* has degree $\ell_1 + 1$, it is complete to Y_1. Hence every vertex $z \in Y_1$ satisfies $d(z) \geq \ell_1 + 2$, and so $y^* \in Y_2$. Let $\overline{y}$ be the vertex in $Y_2 \setminus \{y^*\}$. Since $d(y^*) = \ell_1 + 1$, y^* has only one neighbor in X, and all other vertices in X are adjacent to $\overline{y}$, so $d(\overline{y}) = \ell_1 + k_1 - 1$. It follows that $t = k_1 + 1$ and $d_t = \ell_1 + 1$. If $k_1 = 2$, then $D = X \cup Y_2$, i.e., $X = D \setminus Y_2$, and we are in item 2. If $k_1 \geq 3$, then $d(\overline{y}) \geq \ell_1 + 2$, so $D = X \cup \{y^*\}$, i.e., $X = D \setminus \{y^*\}$, moreover there is only one edge in D and y^* is a vertex of that edge, so we are in item 4.

Now suppose that $\ell_2 = 2$. So $n = k_1 + \ell_1 + 4$, $m_X \leq \ell_1 + 2$, $m_Y \geq \ell_1 + 2$, so we must have $m_X = m_Y = \ell_1 + 2$. Let $Y_2 = \{y, y', z, z'\}$, where the non-adjacent pairs are $\{y, y'\}$ and $\{z, z'\}$. Since $d(x^*) = \ell_1 + 2$, vertex x^* is complete to Y_1 and we may assume that x^* is adjacent to y and z (and not to y' and z'). Hence $d(u) \geq \ell_1 + 3$ holds for each $u \in Y \setminus \{y', z'\}$. Since $d(y^*) = \ell_1 + 2$, we may assume that $y^* = y'$, and y^* is anticomplete to X_1. If also $d(z') = \ell_1 + 2$, then $t = k_1 + 2$, $d_t = \ell_1 + 2$, $D = X \cup \{y', z'\}$, i.e., $X = D \setminus \{y', z'\}$; moreover $y'z'$ is the only edge in D, so we are in item 4. If $d(z') > \ell_1 + 2$, then $t = k_1 + 1$, $d_t = \ell_1 + 2$, and $D = X \cup \{y^*\}$, i.e., $X = D \setminus \{y^*\}$. Note that $\{y, y^*\}$ is the only d_t-pair in G (any vertex u of degree d_t with $u \neq y^*$ is in X, and any vertex v of degree $n - 2$ with $v \neq y$ is in Y_1, and uv is an edge because $d(u) = \ell_1 + 2$). So we are in item 11.

Case 2: $\ell_2 = 0$. So $Y_2 = \emptyset$, $Y = Y_1$, $m_X \leq \ell_1 + 1$ by (1)–(2) and $m_Y \geq \ell_1 - 1 + k_2$ by (3). The inequality $m_X \geq m_Y$ implies $k_2 \leq 2$. If $k_2 = 0$, then G is a split graph, so let us assume that $k_2 > 0$.

Suppose that $k_2 = 1$. So $n = k_1 + \ell_1 + 2$ and $m_Y \geq \ell_1$. If $m_X = \ell_1 + 1$, then x^* is complete to Y and lies in X_2, and then G is a split graph (as $Y \cup \{x^*\}$ is a clique and $X \setminus \{x^*\}$ is a stable set). Hence $m_X \leq \ell_1$, and so $m_X = m_Y = \ell_1$. Vertex y^* has degree ℓ_1 and has one neighbor in X_2, so it is anticomplete to X_1. Hence $d(u) \leq \ell_1 - 1$ for all $u \in X_1$, and $x^* \in X_2$. Let $\overline{x}$ be the vertex in $X_2 \setminus \{x^*\}$. Vertex x^* has $\ell_1 - 1$ neighbors in Y, so $\overline{x}$ has exactly one neighbor in Y (the non-neighbor of x^* in Y). If $\ell_1 = 1$, then G is split (as X_2 is a clique and $X_1 \cup \{y^*\}$ is a stable set). If $\ell_1 = 2$, then $t = n - 1$, $d_t = 2$ and $D' = X_1$,

so we are in item 3. If $\ell_1 \geq 3$, then $t = k_1 + 2$, $d_t = \ell_1$, and $D' = X_1 \cup \{\overline{x}\}$, so $X = D' \cup \{x^*\}$, and $\overline{x}$ is the only 2-vertex in D (because for any $u \in X_1$, $N(u)$ is a clique, and for any $v \in Y \cup \{x^*\}$, $d(v) \geq 3$), so we are in item 5.
Now suppose that $k_2 = 2$. So $n = k_1 + \ell_1 + 4$ and $m_Y \geq \ell_1 + 1$. Hence $m_X = m_Y = \ell_1 + 1$. Let $X_2 = \{x, x', w, w'\}$, where xx' and ww' are edges. Since x^* has degree $\ell_1 + 1$, it is complete to Y and we may assume that $x^* = x'$. Hence x is anticomplete to Y and has degree 1. Since y^* has degree $\ell_1 + 1$ and has two neighbors in X_2, it is anticomplete to X_1. So $d(u) \leq \ell_1 - 1$ for all $u \in X_1$. If also one of w, w' (w', say) has degree $\ell_1 + 1$, then w has degree 1; in that case we have $t = k_1 + 3$, $d_t = \ell_1 + 1$, and $D' = X \setminus \{x^*, w'\}$, i.e., $X = D' \cup \{x^*, w'\}$. We see that xx^* and ww' are the only d_t-edges in G (if a vertex different from x and w has degree 1, then it is in X_1 and its neighbor has degree at least $\ell_1 + 2$), so we are in item 10. If $d(w) \leq \ell_1$ and $d(w') \leq \ell_1$, then $t = k_1 + 3$, $d_t \leq \ell_1$, and $D' = X \setminus \{x^*\}$, i.e., $X = D' \cup \{x^*\}$. We see that xx^* is the only d_t-edge in G, so we are in item 10 again.

Case 3: $k_2 = 1$ and $\ell_2 = 1$. So $n = k_1 + \ell_1 + 4$, $m_X \leq \ell_1 + 2$ and $m_Y \geq \ell_1 + 1$. Let $X_2 = \{x, x'\}$ and $Y_2 = \{y, y'\}$. Suppose that $\ell_1 = 0$. If $k_1 = 0$, then G is split. If $k_1 > 0$ and one vertex in Y_2 is complete to X_1 (vertex y, say), then $D' = X_1 \cup \{y'\}$ and it is easy to see that D' is acceptable, so we are in item 1. If each vertex in Y_2 has a neighbor in X_1, then we find $X_1 \subset D' \subset X$, with $|X_1| = n - 4$, so we are in item 3. Thus we may assume that $\ell_1 > 0$. We distinguish three subcases (a), (b), (c):
(a) $m_X = m_Y = \ell_1 + 2$. Since $d(x^*) = \ell_1 + 2$, vertex x^* is complete to Y and lies in X_2; $x^* = x'$, say. So x has degree 2 and is the only 2-vertex in G (for any $u \in X_1$, $N(u)$ is a clique, and for $v \in Y \cup \{x^*\}$, $d(v) \geq 3$). We find $t = k_1 + 2$, $d_t = \ell_1 + 2$ and $X = D' \cup \{x^*\}$, so we are in item 5.
(b) $m_X = m_Y = \ell_1 + 1$. We may assume that $y^* = y'$ and is anticomplete to X_1. We have $d(z) \geq \ell_1 + 2$ for all $z \in Y_1$. It follows that $t = k_1 + 3$ and $d_t = \ell_1 + 1$. If $d(y) \geq \ell_1 + 2$, then $D = X \cup \{y'\}$, i.e., $X = D \setminus \{y'\}$, and $\{x, x', y'\}$ induces a component in D (a P_3, and the only component of size at least 2), so we are in item 4. If $d(y) = \ell_1 + 1$, then $D = X \cup \{y, y'\}$ and $\{x, x', y, y'\}$ induces a component in D (a P_4), so we are in item 4 again.
(c) $m_X = \ell_1 + 2$ and $m_Y = \ell_1 + 1$. Since $d(x^*) = \ell_1 + 2$, vertex x^* is complete to Y and lies in X_2; $x^* = x'$, say. So x has degree 2. Since $d(y^*) = \ell_1 + 1$, y^* lies in Y_2, $y^* = y'$, say, and is anticomplete to X_1, and so y is complete to X_1. Hence $d(y) = k_1 + \ell_1 + 1$. If $k_1 = 0$, we find $t = 3$, $d_t = \ell_1 + 1$ and $D = \{x, y, y^*\}$; moreover we may assume up to symmetry that the edges between X_2 and Y_2 are xy and x^*y^*, and we observe that $\{x, y\}$ is acceptable, so we are in item 2. Now let $k_1 \geq 1$. Note that x is the unique 2-vertex in G (for all $u \in X_1 \cup \{y^*\}$, $N(u)$ is a clique, and for all $v \in Y_1 \cup \{x^*, y\}$, $d(v) \geq 3$). We find $t = k_1 + 2$, $d_t = \ell_1 + 1$ and $D = (X \setminus \{x^*\}) \cup \{y^*\}$, i.e., $X = (D \setminus \{y^*\}) \cup \{x^*\}$. If the edges between X_2 and Y_2 are xy^* and x^*y, then x^*, y^* are the two neighbors of x, and we are in item 5. So suppose that the edges between X_2 and Y_2 are xy and x^*y^*. If X_1 contains no vertex of degree $\ell_1 + 1$ $(= d_t)$, then $X = D' \cup \{x^*\}$ and we are in item 5. So suppose that X_1 contains a vertex of degree $\ell_1 + 1$. Every such vertex is complete to $Y_1 \cup \{y\}$, and it follows that x^* and possibly y are the only vertices of degree $\ell_1 + 2$ $(= d_t + 1)$, moreover if $d(y) = \ell_1 + 2$ then y has only one neighbor of degree $\ell_1 + 1$. So we are in item 5.

Case 4: $k_2 = 1$ and $\ell_2 = 2$. We have $n = k_1 + \ell_1 + 6$, $m_X \leq \ell_1 + 3$ and $m_Y \geq \ell_1 + 3$. Hence $m_X = m_Y = \ell_1 + 3$. We have $d(u) \leq \ell_1 + 2$ for all $u \in X_1$ and $d(v) \geq \ell_1 + 4$ for all $v \in Y_1$, so $x^* \in X_2$ and $y^* \in Y_2$. It follows that $t = k_1 + 3$ and $d_t = \ell_1 + 3$. We observe that $D' = X_1$ or $D' = X \setminus \{x^*\}$, so $X = D' \cup X_2$; moreover, the vertex in $X_2 \setminus \{x^*\}$ is a 3-vertex, and all 3-vertices are in $X_2 \cup Y_2$, so we are in item 9.

Case 5: $k_2 = 2$ and $\ell_2 = 1$. Let $X_2 = \{x, x', w, w'\}$ with edges xx' and ww'. We have $n = k_1 + \ell_1 + 6$, $m_X \leq \ell_1 + 2$ and $m_Y \geq \ell_1 + 2$. Hence $m_X = m_Y = \ell_1 + 2$. We have $d(u) \leq \ell_1 + 1$ for all $u \in X_1$ and $d(v) \geq \ell_1 + 3$ for all $v \in Y_1$, so $x^* \in X_2$ ($x^* = x'$, say) and $y^* \in Y_2$; moreover, x^* is complete to Y_1, so $d(x) = 2$, and y^* is anticomplete to X_1. It follows that $t = k_1 + 4$ and $d_t = \ell_1 + 2$. If $\ell_1 = 0$, then either $k_1 > 0$ and $D = X \cup \{y^*\}$, i.e., $X = D \setminus \{y^*\}$, and we are in item 7, or $k_1 = 0$ and we are in item 8 (G is a C_6). Now let $\ell_1 \geq 1$. We observe that $X_1 \subseteq D' \subseteq X \setminus \{x^*\}$. If each of w, w' has a neighbor in Y_1, then $X = D' \cup \{x, x^*\}$ and we are in item 5. If one of w, w' is anticomplete to Y_1 (and consequently the other is complete to Y_1), then G has exactly two 2-vertices (x and one of w, w') and we are in item 6. This completes the proof of the theorem. □

The algorithm builds every possible candidate and checks whether it is acceptable. If any candidate is acceptable, then G is a doubled graph; else it is not. We observe that in each item of Theorem 2.2 there is a fixed number of candidates. In total there are at most 38 candidates. Moreover, in each item the candidates can be found in linear time. In item 1 this can be done as explained before the theorem. In all other items, it is easy to see that checking the condition and finding the relevant elements (component of D, 2-vertices, etc) can be done in linear time. So the total time complexity of the algorithm is linear.

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