Les cahiers Leibniz

Construction of snarks with total chromatic number 5

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Construction of snarks with total chromatic number 5[1]

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Abstract

A k-total-coloring of G is an assignment of k colors to the edges and vertices of G, so that adjacent or incident elements have different colors. The total chromatic number of G, denoted by $\chi_T(G)$, is the least k for which G has a k-total-coloring. It was proved by Rosenfeld that the total chromatic number of a cubic graph is either 4 or 5. Cubic graphs with $\chi_T = 4$ are said to be Type 1, and cubic graphs with $\chi_T = 5$ are said to be Type 2.

Snarks are cyclically 4-edge-connected cubic graphs that do not allow a 3-edge-coloring. In 2003, Cavicchioli et al. asked for a Type 2 snark with girth at least 5, but as neither Type 2 snarks nor Type 2 cubic graphs with girth at least 5 are known, this is taking two steps at once and the two requirements of being a snark and having girth at least 5 should better be treated independently. In this paper we will show that the property of being a snark can be combined with being Type 2. We will give a construction that gives Type 2 snarks for each even vertex number $n \geq 40$.

We will also give the result of a computer search showing that among all Type 2 cubic graphs on up to 32 vertices, all but three contain an induced chordless cycle of length 4. These three exceptions contain triangles. The question of the existence of a Type 2 cubic graph with girth at least 5 remains open.

Keywords: snark, total coloring, edge-coloring

1 Introduction

In 1880, Tait [26] proved that the Four-Color Conjecture is equivalent to the statement that every planar bridgeless cubic graph has chromatic index 3. The search for counterexamples to the Four-Color Conjecture motivated the study of cubic graphs with chromatic index 4. Based on the poem by Lewis

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Carroll "The Hunting of the Snark", Gardner [15] introduced the name snark for nontrivial cubic graphs that are not 3-edge-colorable, without giving a precise definition for "nontrivial".

The importance of these graphs arises also from the fact that several well known conjectures would have snarks as minimal counterexamples [9], among them Tutte's 5-Flow Conjecture [27], the 1-Factor Double Cover Conjecture [14], and the Cycle Double Cover Conjecture [25, 23]. In [4] snarks prove their potential as counterexamples by refuting 8 published conjectures.

The Petersen graph is the smallest and earliest known snark. It is known that there are no snarks of order 12, 14 or 16 (see for example [12, 13]). In [18] Isaacs introduced the dot product, a famous operation used for constructing infinitely many snarks, and defined the Flower snark family. The Blanuša snark [2] of order 18 is constructed using the dot product of two copies of the Petersen graph, and Preissmann [20] proved that there are only two snarks of order 18. There are snarks of any even order bigger than 18.

In [8] Cavicchioli et al. reported that their extensive computer study of snarks shows that all snarks of girth at least 5 and with less than 30 vertices are Type 1, and asked for the smallest order of a snark of girth at least 5 that is Type 2. Later on Brinkmann et al. [4] have shown that this order should be at least 38. In 2011, it was proved that all members of the infinite families of Flower and Goldberg snarks are Type 1 [7]. Recently, it has been proved that all members of another two infinite families of snarks are Type 1 [10].

The question of Cavicchioli et al. has two requirements for the Type 2 cubic graphs and so far for none of these two requirements a Type 2 cubic graph is known. Therefore one should split the question in two parts in order to investigate what the real obstruction is that makes these graphs so hard to find (if they exist at all): being snarks or having girth at least 5.

- Does there exist a Type 2 cubic graph with girth at least 5?
- Does there exist a Type 2 snark?

In this paper we will give a positive answer to the last question and a construction that provides an infinite sequence of Type 2 snarks. The first question remains unsolved, but our computational results show that a possible Type 2 cubic graph with girth at least 5 must have at least 34 vertices. Furthermore it seems that, even for girth 3, Type 2 cubic graphs with no induced chordless cycles of length 4 are difficult to find.

2 Definitions

In this section, we give some definitions and notation that will be used in this paper.

A *semi-graph* is a 3-tuple $G = (V(G), E(G), S(G))$ where $V(G)$ is the set of vertices of G, $E(G)$ is a set of edges having two distinct endpoints in $V(G)$, and $S(G)$ is a set of *semi-edges* having one endpoint in $V(G)$. When there is no chance of ambiguity, we will simply write V, E or S.

We will write edges having endpoints v and w shortly as vw and semi-edges having endpoint v as $v\cdot$. When vertex v is an endpoint of $e \in E \cup S$ we will say that v and e are *incident*. Two elements of $E \cup S$ incident to the same vertex, respectively two vertices incident to the same edge, will be called *adjacent*.

A *graph* G is a semi-graph with an empty set of semi-edges. In that case we can write $G = (V, E)$. Given a semi-graph $G = (V, E, S)$, we call the graph (V, E) the *underlying graph of* G. Notice that we can also associate to G the graph G' obtained from G as follows: for each semi-edge $s = v\cdot$ of G, create a new vertex x_s and replace s by a new edge vx_s.

All definitions given below for semi-graphs, that do not require the existence of semi-edges, are also valid for graphs.

Let $G = (V, E, S)$ be a semi-graph.

The *degree* $d(v)$ of a vertex v of G is the number of elements of $E \cup S$ that are incident to v. We say that G is *d-regular* if the degree of each vertex is equal to d. In this paper we are mainly interested in 3-regular graphs and semi-graphs, also called respectively *cubic graphs* and *cubic semi-graphs*. Given a graph G of maximum degree 3, the semi-graph obtained from G by adding $3 - d(v)$ semi-edges with endpoint v, for each vertex v of G, is called *the cubic semi-graph generated by* G and is denoted by *s-G*.

For $k \in \mathbb{N}$, a *k-vertex-coloring* of G is a map $C^V\colon V \to \{1, 2, ..., k\}$, such that $C^V(x) \neq C^V(y)$ whenever x and y are two adjacent vertices. The *chromatic number* of G, denoted by $\chi(G)$, is the least k for which G has a k-vertex-coloring.

Similarly, a *k-edge-coloring* of G is a map $C\colon E \cup S \to \{1, 2, ..., k\}$, such that $C(e) \neq C(f)$ whenever e and f are adjacent elements of $E \cup S$. The *chromatic index* of G, denoted by $\chi'(G)$, is the least k for which G has a k-edge-coloring. By Vizing's theorem [28] we have that $\chi'(G)$ is equal to either $\Delta(G)$ or to $\Delta(G) + 1$, where $\Delta(G)$ is the maximum degree of the vertices of G. If $\chi'(G) = \Delta(G)$, then G is said to be *Class 1*, otherwise G is said to be *Class 2*.

A *k-total-coloring* of G is a map $C^T\colon V \cup E \cup S \to \{1, 2, ..., k\}$, such that

- **(a)** $C^T|_V$ is a vertex-coloring,
- **(b)**$C^T|_{E\cup S}$ is an edge-coloring,
- **(c)** $C^T(e) \neq C^T(v)$ whenever $e \in E \cup S$, $v \in V$, and e is incident to v.

The *total chromatic number* of G, denoted by $\chi_T(G)$, is the least k for which G has a k-total-coloring. Clearly $\chi_T(G) \geq \Delta(G) + 1$. The Total Coloring Conjecture [28, 1] claims that $\chi_T(G) \leq \Delta + 2$. If $\chi_T(G) = \Delta(G) + 1$, then G is said to be *Type 1*, and if $\chi_T(G) = \Delta(G) + 2$, then G is said to be *Type 2*.

The Total Coloring Conjecture has been proved for cubic graphs [22]. Thus, a cubic graph is either Type 1 (its total chromatic number is equal to 4) or Type 2 (its total chromatic number is equal to 5).

We will show now that $(k+1)$-total-colorings of k-regular semi-graphs are characterized by some particular $(k+1)$-edge-colorings, so that in a later proof we do not have to care about vertex-coloring.

Definition 1. *A $(k+1)$-edge-coloring C of a k-regular semi-graph $G = (V, E, S)$ is called a* **strong $(k+1)$-edge-coloring** *if for each edge $vw \in E$ we have*
$|\{C(e)|e \in E \cup S, e \text{ incident to } v \text{ or } w\}| = k + 1.$

Equivalently a strong $(k+1)$-edge-coloring of a k-regular semi-graph is a $(k+1)$-edge-coloring such that for each edge vw the color not used for the elements of $E \cup S$ incident to v must be used for an element incident to w. A strong $(k+1)$-edge-coloring has the property that it is not possible to obtain another $(k+1)$-edge-coloring by changing the color of a single edge.

Lemma 1. *Let $G = (V, E, S)$ be a k-regular semi-graph.*

Each strong $(k+1)$-edge-coloring C of G can be extended to a $(k+1)$-total-coloring C^T with $C^T|_{E\cup S} = C$, and, for each $(k+1)$-total-coloring C^T of G, $C^T|_{E\cup S}$ is a strong $(k+1)$-edge-coloring.

This implies:

There exists a $(k+1)$-total-coloring C^T of G if and only if there exists a strong $(k+1)$-edge-coloring C of G.

Proof. If C is a $(k+1)$-edge-coloring of G then, for any vertex v, all k elements of $E \cup S$ incident to v are colored differently. So, there exists a $(k+1)$-total-coloring C^T of G such that $C = C^T|_{E\cup S}$ if and only if the function $C^V : V \to \{1, 2, ..., k+1\}$, such that $C^V(v)$ is equal to the unique color not used by C for the elements of $E \cup S$ incident to v, is a $(k+1)$-vertex-coloring of G (thus extending C to a total coloring). □

By Lemma 1, a cubic semi-graph G is of Type 1 if and only if there exists a strong 4-edge-coloring of G. This will be used later to prove that some semi-graphs are not of Type 1 and in a computer program computing the type of cubic graphs.

The following Lemma is well known and useful.

Lemma 2. *Parity Lemma (Blanuša, 1946 [2] - Descartes, 1948 [11]) Let G be a cubic semi-graph containing exactly k semi-edges, C be a 3-edge-coloring of G, and k_1, k_2, k_3 be the number of semi-edges of G colored respectively 1, 2, 3 by C. Then*

$$k_1 \equiv k_2 \equiv k_3 \equiv k \bmod (2).$$

A *triangle* is a graph consisting of a cycle of length 3 (or equivalently a complete graph on three vertices), a *square* is a graph consisting of a chordless cycle of length 4, and an *s-square* is a cubic semi-graph generated by a square. A K_4 is a complete graph on four vertices.

In the rest of this section, $G = (V, E)$ will be assumed to be a graph.

The *girth of* G is the minimum length of a cycle contained in G, or if G has no cycle, it is defined to be infinity. A *subgraph* of G is any graph $G' = (V', E')$ such that $V' \subseteq V$ and $E' \subseteq E$. An *induced subgraph* of G is any subgraph $G' = (V', E')$ such that E' is equal to the set of edges of G with both endpoints in V'. We will denote it by $G[V']$. Given a graph H, G will be called *H-free* if none of its induced subgraphs are isomorphic to H.

Let A be a proper subset of V. We denote by $\delta_G(A)$ the set of edges of G with one endpoint in A and the other endpoint in $V \setminus A$. A subset F of edges of G is an *edge cutset* if there exists a proper subset A of V such that $F = \omega_G(A)$ and we will then say that F *is induced by* A. If each of $G[A]$ and $G[V \setminus A]$ has at least one cycle then $\omega_G(A)$ is said to be a *c-cutset*. We say that G is *cyclically k-edge-connected* if it has not c-cutset of cardinality smaller than k. If G has at least one c-cutset, the *cyclic-edge-connectivity* of G is the smallest cardinality of a c-cutset of G, else it is set to infinity. It is useful to notice the following.

Remark 1. *A cubic graph is cyclically 4-edge-connected if and only if each of its edge cutsets of cardinality smaller than 4 is induced by a single vertex.*

In this paper, a *snark* is a cyclically 4-edge-connected Class 2 cubic graph. Some authors use other definitions: in [24] snarks are just Class 2 cubic graphs, in [29] Zhang adds the condition of being bridgeless, but most commonly, snarks are defined as cyclically 4-edge-connected Class 2 cubic graphs, with sometimes the additional property to be square-free (notice that for a cubic graph with more than four vertices, being cyclically 4-edge-connected obviously implies being triangle-free). The name “snark” comes from Martin Gardner [15] who, in its popular chronicle of Scientific American, proposed it as a substitute for “Nontrivial trivalent graphs which are not Tait-colorable” used by Isaacs [18]. Without going into details, the “triviality”of c-cutsets of size less than 4 is not the same as the one of c-cutsets induced by squares (see [21]), and forbidding squares was left by Isaacs and Gardner as “optional”. The power of the notion of snark comes from the fact that several important conjectures in graph theory would be proved if shown valid for snarks. For some conjectures, but not all, this remains true when adding the square-free condition, or even stronger bounds for the girth (see [29] page 253).

With respect to the Type of a cubic graph, neither a square nor a small cyclic-edge-connectivity seem to be “trivial”: for cyclic-edge-connectivity 2 or 3 there exist examples of cubic graphs of each possible Class and Type and for cyclic-edge-connectivity 1 there exist examples of cubic Class 2 graphs of each Type [10].

2.1 The principle of the construction

The idea of our construction of Type 2 snarks is based on the following fact.

Remark 2. *A graph G containing a Class 2 (resp. Type 2) subgraph with maximum degree $\Delta(G)$ is Class 2 (resp. Type 2).*

So a way to build a Type 2 snark is to make a cyclically 4-edge-connected cubic graph from a graph of maximum degree 3 which is Class 2 and a graph of maximum degree 3 which is Type 2. This is what is explained in this subsection.

Definition 2. *A* **brick** *is a cubic semi-graph $B = (V, E, S)$ with exactly four pairwise non-adjacent semi-edges and whose underlying graph (V, E) is a subgraph of some cyclically 4-edge-connected cubic graph.*

Given two disjoint bricks $B = (V, E, S)$ and $B' = (V', E', S')$, any graph $G = (V \cup V', E \cup E' \cup E'')$ with E'' being a set of four disjoint edges xy with $x\cdot \in S$ and $y\cdot \in S'$, is called **a junction of B and B'**.

The following properties of bricks are useful.

Lemma 3. **(a)** *The underlying graph of a brick is bridgeless.*

(b) *Any junction of two bricks is a cyclically 4-edge-connected graph.*

(c) *An s-square is a brick. No brick has less than four vertices.*

Proof. (a) Let G_B be the underlying graph of a brick B. Then G_B has four vertices of degree 2 and all others have degree 3. Since B is a brick, G_B is the subgraph of some cyclically 4-edge-connected cubic graph H.

If G_B had a bridge, the fact that G_B has minimum degree 2 implies that after the removal of the bridge both components would contain a cycle. At least one of the components would contain at most two vertices of degree 2 – but in H this component would be attached to the rest by at most three edges – so H would have a c-cutset of size at most 3 – a contradiction.

(b) Let $B = (V, E, S)$ and $B' = (V', E', S')$ be disjoint bricks and G be any graph obtained by a junction of B and B'. If G would contain a c-cutset of cardinality at most 3, then one of B, B' would contain at least two edges of this c-cutset, because otherwise, by (a), both B and B' would be connected after removing the edges, and also connected to each other by at least one edge. So assume that B contains at least two edges of the c-cutset, implying that B' contains at most one. But then, as B' stays connected by (a), one of the cyclic components after removal of the c-cutset must be entirely in B – so this component could be split off by removing at most three edges in any cubic graph containing the underlying graph of B – a contradiction.

(c) The proof is immediate as there are cyclically 4-edge-connected cubic graphs that have squares as subgraphs, as for example K_4 or the 3-dimensional cube. Furthermore, by definition a brick should have at least four vertices. □

As an immediate consequence of Lemma 3 we obtain the following characterization of a brick.

Remark 3. *A cubic semi-graph $B = (V, E, S)$ with exactly four pairwise non-adjacent semi-edges is a brick if and only if any junction of B and an s-square is cyclically 4-edge-connected.*

Furthermore, Lemma 3 and Remark 2 provide directly the following theorem.

Theorem 1. *A graph obtained by a junction of a Class 2 brick and a Type 2 brick is a Type 2 snark.*

So it is enough to discover one brick of Class 2 and one brick of Type 2 in order to be able build a Type 2 snark. Before to consider these problems, we will study some more definitions and properties of bricks.

Definition 3. *Let G be a cyclically 4-edge-connected cubic graph.*

Removing two non-adjacent edges of G, one obtains a graph generating a cubic semi-graph B with exactly four pairwise non-adjacent semi-edges, *which is by definition a brick. Such a brick is called a* **direct-brick of** G*. Two non-adjacent vertices of a direct-brick* B *of* G *that are adjacent in* G *are called a* **pair of** B *(with respect to* G*). Two semi-edges incident to vertices of the same pair of* B *are also called a* **pair**. *By definition, a direct-brick of* G *contains two pairs of semi-edges.*

Removing two adjacent vertices of G *and all their incident edges, if* G *has at least six vertices, one obtains a graph generating a cubic semi-graph* B *with exactly four pairwise non-adjacent semi-edges, which is by definition a brick. Such a brick is called an* **edge-brick of** G*. Two vertices of* B *that are in* G *adjacent to the same vertex not in* B *are called a* **pair of** B *(with respect to* G*). Two semi-edges incident to vertices of the same pair of* B *are also called a* **pair**. *By definition, an edge-brick of* G *contains two pairs of semi-edges.*

Notice that there exist bricks that are not direct-bricks. One such brick is shown on Figure 1.

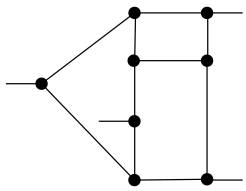

Figure 1: A brick that is not a direct-brick.

On another hand, we have the following theorem.

Theorem 2. *All bricks are edge-bricks.*

Proof. Let $B = (V, E, S)$ be a brick and $B' = (V, E)$ be the underlying graph of B. By the definition of a brick, B' has a set T of four vertices of degree 2 and all other vertices have degree 3. Our goal is to show that after adding to B' two new vertices x and y and an edge xy, there is a way to connect x and y to the vertices of T so that the resulting graph G is cubic and cyclically 4-edge-connected.

First let us assume that B is an s-square. Then B is an edge-brick of the cyclically 4-edge-connected cubic graph obtained by connecting each of x and y to non adjacent vertices of T.

So, from now on, we may assume that B is not an s-square. The way we then define G depends on the existence of a particular subset of vertices of B' :

- if there no subset of vertices of B' containing exactly two vertices of T and inducing an edge-cutset of size 2 then set G as any cubic graph obtained by adding four edges between xy and T.

- else choose $X \subseteq V(B')$ such that $|\delta_{B'}(X)| = 2$ and $|X \cap T| = 2$, and set G as any cubic graph obtained by adding four edges between xy and T in such a way that each of x and y gets a neighbour in X.

In order to show that G is a cyclically 4-edge-connected graph we will need the following lemmas.

Lemma 4. *If W is a non empty subset of V containing at most two vertices of T, then either :*

- $|\delta_{B'}(W)| = 2$ *and* $|W| = 1 = |W \cap T|$,
- $|\delta_{B'}(W)| = 2$ *and* $|W \cap T| = 2$;
- $|\delta_{B'}(W)| = 3$, $|W| = 1$ *and* $|W \cap T| = 0$,
- $|\delta_{B'}(W)| = 3$, $|W| \geq 2$ *and* $|W \cap T| \geq 1$,
- $|\delta_{B'}(W)| \geq 4$.

As for Lemma 3(a), the validity of Lemma 4 follows easily from the fact that B' is (by definition) a subgraph of some cyclically 4-edge-connected cubic graph.

Lemma 5. *If W is a subset of vertices of G that contains at least one vertex of $T \cup \{x, y\}$ but not all, then $|\delta_G(W)| \geq |\delta_{B'}(W \setminus \{x, y\})| + 1$.*

The validity of Lemma 5 comes from the fact that $\{x, y\} \cup T$ induces in G a connected subgraph whose no edge is in B'.

Lemma 6. *If W is a subset of vertices of G such that $|W \cap T| = 2$ and each of x and y has a neighbor in W, then $|\delta_G(W)| \geq |\delta_{B'}(W \setminus \{x, y\})| + 2$.*

The validity of Lemma 6 comes the following facts : the subgraph of G induced by $\{x, y\} \cup T$ contains a path joining the neighbors of x that are in T and a path joining the neighbors of y that are in T, these paths are disjoint, no edge of these paths is in B'.

Lemma 7. *If W is a subset of vertices of G such that $|W \cap (T \cup \{x, y\})| \geq 1$, $|W \cap T| \leq 2$, and $|\delta_G(W)| \leq 3$, then either $|W| = 1$, or $|\delta_{B'}(W \setminus \{x, y\})| = 2 = |W \cap T|$.*

Proof of Lemma 7. If $W \subseteq \{x, y\}$ then since G is cubic and $|\delta_G(W)| \leq 3$ we get that either $W = \{x\}$ or $W = \{y\}$, so $|W| = 1$.

Else, from Lemmas 4 and 5 and our assumption, we have that $2 \leq |\delta_{B'}(W \setminus \{x, y\})| \leq |\delta_G(W)| - 1 \leq 2$, which implies $|\delta_{B'}(W \setminus \{x, y\})| = 2$. If $|W \cap T| \neq 2$ then, by Lemma 4, we have that $|W \setminus \{x, y\}| = |W \cap T| = 1$: then $|W| \leq 3$ and by assumption $|\delta_G(W)| \leq 3$. As G is cubic, this is possible if and only if either $|W| = 1$ or W induces a triangle in G. As any vertex in T is adjacent to exactly one vertex in $\{x, y\}$ this last case is impossible.
This ends the proof of Lemma 7.

Assume that G is not cyclically-4-edge-connected. We will show that this leads to a contradiction. Indeed, then there exists $W \subset V(G)$ such that $|W| \geq 2$, $|V(G) \setminus W| \geq 2$, and $|\delta_G(W)| \leq 3$. We can assume that W contains at most two vertices of T (else we may consider its complement).

If W contains no vertex of $T \cup \{x, y\}$ then $|\delta_{B'}(W)| = |\delta_G(W)| \leq 3$. So, by Lemma 4, W contains only one vertex, and we get a contradiction.

If W contains at least one vertex of $T \cup \{x, y\}$, then by Lemma 7 and the assumption that $|W| \geq 2$, we get that $|\delta_{B'}(W \setminus \{x, y\})| = 2$ and that W contains exactly two vertices of T, say u and v. If u and v have distinct neighbors in $\{x, y\}$ then by Lemma 6 we would have $|\delta_G(W)| \geq 4$ which contradicts our assumption. So, without loss of generality, we can assume that u and v are both adjacent to x, and then we must have $x \in W, y \notin W$ (else $|\delta_G(W)| \geq 4$, a contradiction again). This means that when we built G we did not choose $W \setminus \{x\}$ but another subset X such that $X \subseteq V(B')$, $|\delta_{B'}(X)| = 2$, $|X \cap T| = 2$. Then, by definition of G we have $|X \cap \{u, v\}| = 1$. Let w, t be the two neighbors of y in G that are in T. Without loss of generality, we can assume that $u, w \in X$ and $v, t \notin X$. So we have $|\delta_G(X)| = 4$ and $|\delta_G(W)| = 3$. Let $V_u = X \cap W$, $V_w = X \setminus V_u$, $V_v = W \setminus V_u$ and $V_t = V(G) \setminus (X \cup W)$. We have that $u \in V_u$, $w \in V_w$, $\{v, x\} \subseteq V_v$, $\{t, y\} \subseteq V_t$, so each of V_u, V_v, V_w, V_t contains a distinct vertex of T. By Lemma 7, $|\delta_G(V_v)| \geq 4$ and $|\delta_G(V_t)| \geq 4$. By Lemmas 4 and 5 we have $|\delta_G(V_u)| \geq 3$ and $|\delta_G(V_w)| \geq 3$. As $|\delta_G(V_u)| + |\delta_G(V_v)| + |\delta_G(V_w)| + |\delta_G(V_t)| \leq 2(|\delta_G(X)| + |\delta_G(W)|) = 2(4 + 3) = 14$, we get that $|\delta_G(V_u)| = |\delta_G(V_w)| = 3$, $|\delta_G(V_v)| = |\delta_G(V_t)| = 4$ and no edge belongs to both $\delta_G(X)$ and $\delta_G(W)$. Then by Lemma 7, $|V_u| = |V_w| = 1$. We also have that the edges xu and xy belongs to $\delta_G(V_v)$, so $|\delta_{B'}(V_v \setminus \{x, y\})| = 2$ and by Lemma 4, one should have $|V_v \setminus \{x, y\}| = 1$. Similarly $|V_t \setminus \{x, y\}| = 1$. So we get that $V_u = \{u\}$, $V_w = \{w\}$, $V_v = \{v, x\}$, $V_t = \{t, y\}$. From all the constraints above, the only possible additional edges are uw, uv, wt and vt. So B is a square-brick, a contradiction to our assumption. This ends the proof of the theorem. $\square$

3 Finding Class 2 bricks

In this section, we will show how to obtain Class 2 bricks and prove that the smallest ones have 18 vertices.

We will show that Class 2 bricks can be obtained from the dot product of snarks. Before explaining this, we will give some useful properties that were noticed by Isaacs.

Lemma 8. *(Isaacs, 1976 [18]) Let G be a snark and let B be a direct-brick of G. In every 3-edge-coloring of B, two semi-edges that form a pair with respect to G get distinct colors.*

Proof. Immediate from the fact that G is Class 2 and the Parity Lemma. □

Lemma 9. *(Isaacs, 1976 [18]) Let G be a snark and let B be an edge-brick of G. In every 3-edge-coloring of B, the semi-edges that are in a pair with respect to G get the same color.*

Proof. Immediate from the fact that G is Class 2 and the Parity Lemma. □

Definition 4. *Let S_1, S_2 be disjoint snarks, $B_1 = (V_1, E_1, S_1)$ a direct-brick of S_1 with vertex pairs r_1, r_2 and s_1, s_2 and $B_2 = (V_2, E_2, S_2)$ an edge-brick of S_2 with vertex pairs x_1, x_2 and y_1, y_2.*

Then $D = (V_1 \cup V_2, E_1 \cup E_2 \cup \{r_1x_1, r_2x_2, s_1y_1, s_2y_2\})$ is called a **dot product of** *S_1, S_2.*

So, a dot product is a pair-to-pair junction of one direct-brick and one edge-brick, both bricks arising from snarks. The dot product is depicted in Figure 2.

Theorem 3. *(Isaacs, 1976 [18]) Any dot product of two snarks is a snark.*

Proof. Let S be the result of a dot product of two snarks. By definition, S is a junction of B_1 and B_2, where B_1 is a direct-brick of a snark S_1, and B_2 is an edge-brick of a snark S_2. Then by Lemma 3, S is cyclically 4-edge-connected. Any 3-edge-coloring of S would induce 3-edge-colorings of B_1 and B_2 that coincide on semi-edges forming an edge in S. As S is a pair-to-pair junction of B_1 and B_2, Properties 8 and 9 give contradicting results – depending on whether the colors are examined from B_1 or B_2. So S is Class 2. □

Definition 5. *Let S_1, S_2 be two disjoint snarks, B_1 be a direct-brick of S_1 with vertex pairs r_1, r_2 and s_1, s_2 and B_2 be an edge-brick of S_2 with vertex pairs x_1, x_2 and y_1, y_2.*

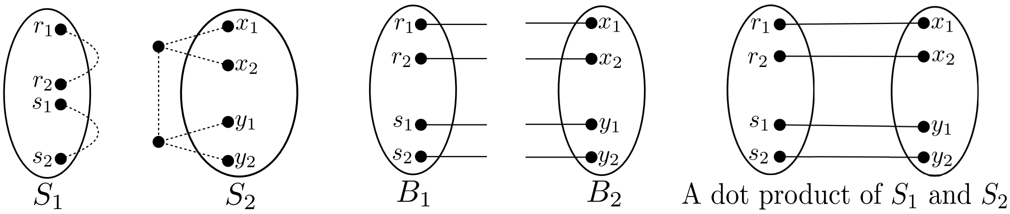

Figure 2: An illustration of the dot product of S_1 and S_2.

Furthermore, let $D = (V, E)$ be the dot product of S_1, S_2 with edges $\{r_1x_1, r_2x_2, s_1y_1, s_2y_2\}$ and z a neighbor of r_2 in D.

Then $Z = (V, E', S')$ with $E' = E \setminus \{r_1x_1, r_2z\}$ and $S' = \{r_1\cdot, x_1\cdot, r_2\cdot, z\cdot\}$ is called a **semi-dot product** *of S_1, S_2.*

Lemma 10. *A semi-dot product Z of two snarks is a Class 2 brick.*

Proof. By definition Z is a direct-brick of a dot product of two snarks. What remains to be shown is that it is Class 2. We use the notation as in Definition 5.

If $z = x_2$ the result follows exactly like in the proof of Theorem 3 as the contradiction in that proof already follows from the connections between one of the two sets of paired vertices.

So assume that $z \neq x_2$ and that C is a 3-edge-coloring of Z. C induces a 3-edge-coloring of B_2, so by Lemma 9: $C(r_2x_2) = C(x_1\cdot)$ which is w.l.o.g. equal to 1. Then we can fix $C(r_2\cdot) = 2$ and the Parity Lemma and Lemma 8 applied to Z give $C(r_1\cdot) = 2$ and $C(z\cdot) = 1$.

But then C' defined as $C' = C$ on $E(B_1) \setminus \{r_2z\}$, $C'(r_2z) = 1$, $C'(s_1\cdot) = C'(s_2\cdot) = C(s_1y_1)$ and $C'(r_1\cdot) = C'(r_2\cdot) = 2$ is a 3-edge-coloring of B_1 contradicting Lemma 8. □

Theorem 4. *The smallest Class 2 bricks have 18 vertices and there are 5 of them.*

Proof. The Petersen graph is the smallest snark, so the smallest Class 2 bricks that can be obtained by the semi-dot product come from an edge-brick and a direct brick of the Petersen graph and have 18 vertices. Due to the high symmetry of the Petersen graph, there are only two direct-bricks and one edge-brick of the Petersen graph. Applying it in every possible way, the construction gives 5 non-isomorphic Class 2 bricks of order 18.

Using a computer and exhaustively enumerating all cubic semi-graphs with exactly four pairwise non-adjacent semi-edges up to 18 vertices and filtering them for Class 2 bricks, we checked that these five bricks constitute in fact the set of all Class 2 bricks of order at most 18. Figure 3 shows one of these graphs. □

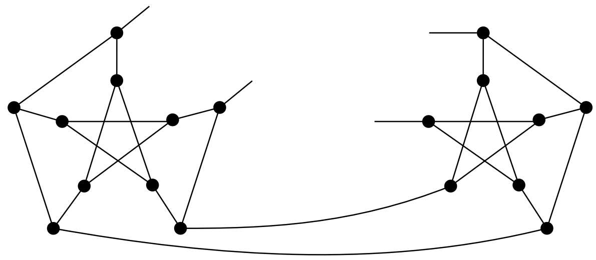

Figure 3: One of the smallest Class 2 bricks.

Note that some direct-bricks of Loupekhine snarks [19] can also be shown to be Class 2. The smallest such brick is the critical Class 2 graph of order 22 with maximum degree 3 due to Goldberg [16].

4 Finding Type 2 bricks

Type 2 bricks were not easy to find. From Hamilton and Hilton [17] it was known that a Type 2 brick would have at least 18 vertices. A computer search determined that the smallest Type 2 brick has 22 vertices. It is depicted in Figure 4 and we will denote it B^*. We find it worth explaining why this graph is a brick of Type 2.

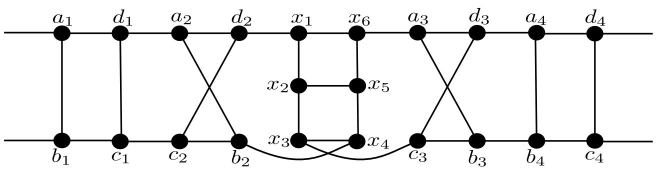

Figure 4: The Type 2 brick B^*.

Lemma 11. *In every 4-total-coloring of an s-square, semi-edges incident to adjacent vertices get distinct colors.*

Proof. Let us consider an s-square S with edges ab, bc, cd, da (Figure 5). Assume that the lemma is not true. Then there exists a 4-total-coloring C^T of S such that $a\cdot$ and $b\cdot$ are colored the same. By Lemma 1 applied to edge ab, we can assume w.l.o.g. that $C^T(a\cdot) = 1$, $C^T(b\cdot) = 1$, $C^T(ab) = 2$, $C^T(ad) = 3$, and $C^T(bc) = 4$. Moreover, applying Lemma 1 to edges ad and

bc, we get $C^T(d\cdot) = 4$ and $C^T(c\cdot) = 3$. But then, edge cd cannot satisfy Lemma 1, and we get a contradiction. □

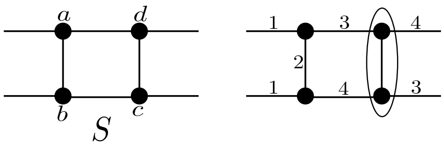

Figure 5: An attempt to find a strong-edge-coloring of S.

Let us call the graph consisting of two squares sharing an edge, a *domino*, and the cubic semi-graph generated by a domino an *s-domino*. Two vertices at maximum distance in an s-domino are called *opposite*. An s-domino contains two pairs of opposite vertices.

Lemma 12. *Every 4-total-coloring C^T of an s-domino is such that, for one pair of opposite vertices x and y, the following holds: $C^T(x) = C^T(y)$ and $C^T(x\cdot) \neq C^T(y\cdot)$.*

Proof. Let us consider an s-domino D with edges x_1x_2, x_2x_3, x_3x_4, x_4x_5, x_5x_6, x_6x_1, x_2x_5, and let C^T be a 4-total-coloring of D (Figure 6).

By symmetry, by Lemma 1 for edge x_2x_5, and Lemma 11, we can assume w.l.o.g. that $C^T(x_2x_5) = 1$, $C^T(x_1x_2) = C^T(x_5x_4) = 2$, $C^T(x_2x_3) = 3$, and $C^T(x_5x_6) = 4$.

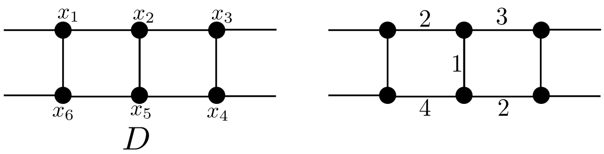

Figure 6: Strong-edge-coloring applied to edge x_2x_5 of D.

By Lemma 1 for edge x_1x_2, and the fact that $C^T(x_5x_6) = 4$, we have that $C^T(x_1\cdot) = 4$; similarly, we must have $C^T(x_4\cdot) = 3$. Lemma 1 for edge x_1x_6 implies that edge x_1x_6 and semi-edge $(x_6\cdot)$ must have colors 1 and 3 (not necessarily in this order). Similarly, Lemma 1 for edge x_3x_4 implies that edge x_3x_4 and semi-edge $(x_3\cdot)$ must have colors 1 and 4.

In any case, x_6 and x_3 receive color 2, so these vertices have the desired property except for the case where $C^T(x_6\cdot) = C^T(x_3\cdot) = 1$. But in this case, we have $C^T(x_3x_4) = 4$ and $C^T(x_5x_6) = 3$, and so vertices x_1 and x_4 have the desired property. □

Theorem 5. *The semi-graph B^* of Figure 4 is a brick of Type 2.*

Proof. Consider the semi-graph B^* in Figure 4: it is a cubic semi-graph with exactly four pairwise non-adjacent semi-edges and adding e.g. the non-adjacent edges a_1d_4 and b_1c_4 we obtain a cyclically 4-edge-connected cubic graph, so B^* is a (direct-) brick. Now, we will prove that it is of Type 2.

Let us remark that vertices x_1, x_2, x_3, x_4, x_5, and x_6 of B^* induce a domino in B^*. Furthermore, the opposite pairs of this domino are connected similarly to the same graph ($B^*[a_1, b_1, c_1, d_1, a_2, b_2, c_2, d_2]$ is isomorphic to $B^*[a_3, b_3, c_3, d_3, a_4, b_4, c_4,$
$d_4]$).

Assume that there exists a 4-total-coloring C^T of B^*. Then, by Lemma 12 and the symmetry of B^*, one can assume w.l.o.g. that $C^T(x_1) = C^T(x_4) = 1$, $C^T(x_1d_2) = 2$, and $C^T(x_4b_2) = 3$.

Now, by Lemma 11 applied to the s-square generated by a_2, b_2, c_2, d_2, edges d_1a_2 and c_1c_2 cannot be colored with colors 2 or 3 by C^T. On the other hand, by Lemma 11 applied to the s-square generated by a_1, b_1, c_1, d_1, edges d_1a_2 and c_1c_2 cannot be colored the same by C^T.

So, we can assume w.l.o.g. that $C^T(d_1a_2) = 1$ and $C^T(c_1c_2) = 4$.

Now, remark that since vertices x_1 and x_4 are colored 1, there is no edge incident to them that is colored 1. So, by Lemma 1 applied to edges x_1d_2 and x_4b_2 and the fact edge d_1a_2 is colored 1, we have that $C^T(d_2c_2) = C^T(b_2c_2) = 1$, which is impossible. So, we get a contradiction and there is no 4-total-coloring of B^*. □

Of course any brick that has the underlying graph of B^* as a subgraph is Type 2. A computer search gave that up to 26 vertices this is the only way to obtain larger Type 2 bricks:

Remark 4. *There is one Type 2 brick with 22 vertices, it is the brick B^* depicted in Figure 4. There are two Type 2 bricks with 24 vertices and there are ten Type 2 bricks with 26 vertices. All these bricks contain the underlying graph of B^* as a subgraph.*

The 22 vertex brick can be downloaded from the graph database HoG [6] by searching for the keyword `Type2_brick_22`.

5 Type 2 snarks

Now, we are able to present the main result of this paper. Since a junction of a Class 2 brick with a Type 2 brick gives a Type 2 snark, we can construct Type 2 snarks.

Theorem 6. *There exist Type 2 snarks for each even order $n \geq 40$.*

Proof. Any junction of the Type 2 brick of order 22 of Figure 4 and a Class 2 brick of order 18 as described in Section 3 (Figure 3) is a Type 2 snark of order 40. Using a computer to construct all possible combinations it was found that up to isomorphism there are 11 such Type 2 snarks of order 40. Figure 7 presents one of them. All graphs can be downloaded from the graph database HoG [6] by searching for the keyword `Type2_Class2_40`.

From any Class 2 or Type 2 brick B it is easy to obtain a new one with two more vertices v and w: delete two semi-edges $x\cdot$, $y\cdot$ and add edges xv, vw, wy and semi-edges $v\cdot$, $w\cdot$. Indeed, the so obtained semi-graph is an edge-brick of the cyclically 4-edge-connected graph obtained by the junction of B and a square; and it contains the underlying graph of B as a subgraph. So there exist snarks of Type 2 for all even orders from 40 on. □

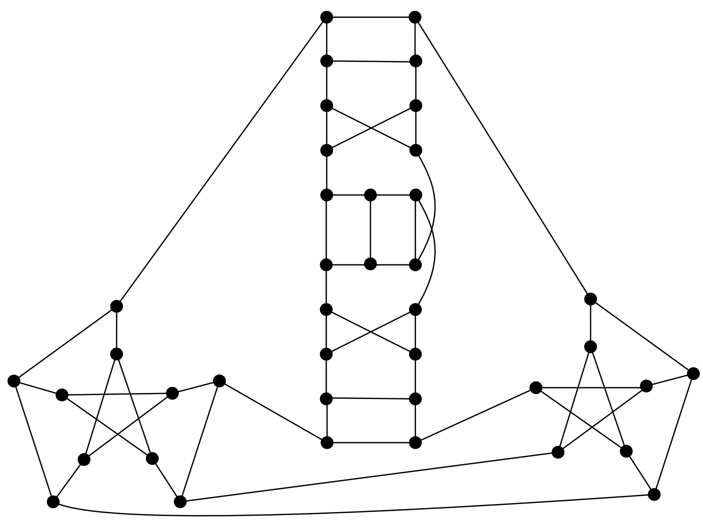

Figure 7: One of the 11 smallest snarks of Type 2 obtained by a junction of the bricks on Figure 3 and Figure 4.

Question 1. *Does there exist a Type 2 snark of order less than 40?*

A computer search gave that all snarks on up to 34 vertices are Type 1, so the only possible orders for which the existence is not yet known are 36 and 38.

6 Checking the programs

For the generation of the cubic graphs, the program *minibaum* described in [3] was used. The snarks were generated by the program *snarkhunter* of which the construction and isomorphism routines are described in [5]. The cubic semi-graphs with exactly four pairwise non-adjacent semi-edges were generated by the program *multigraph* that is based on the techniques of *minibaum*. It was never published, but is used since almost 20 years and during this time the results were compared against the results of various other programs and there was never any discrepancy.

Except for the program testing the property of being a brick, most testing programs are also used since long. They were e.g. also used in [4] where most of the results were compared against independent results. For the largest cases (e.g. square-free cubic graphs on 32 vertices) where the total chromatic number had to be determined, a special program had to be developed. It uses Lemma 1 and searches for strong 4-edge-colorings. For testing purposes the results were compared to the results of the older, well tested and independent program for all cubic graphs up to 24 vertices and for triangle-free cubic graphs on 26 vertices and were found to be in complete agreement.

For testing whether a cubic semi-graph with exactly four pairwise non-adjacent semi-edges is a brick, two programs were written. One checked for cuts and the distribution of semi-edges in the components after removing the cuts, and the other (faster one) used Lemma 3. It did a junction of the cubic semi-graph with four pairwise non-adjacent semi-edges with an s-square and tested the resulting graph for being cyclically 4-edge-connected. The programs were compared for all cubic semi-graphs with four pairwise non-adjacent semi-edges on up to 18 vertices (more than 4.000.000 graphs) and were found to be in complete agreement.

7 Conclusion

In 2003, Cavicchioli et al. [8] reported that their extensive computer study of snarks shows that all snarks of girth at least 5 and with less than 30 vertices are Type 1, and asked for the smallest order of a snark of girth at least 5 that is Type 2. Later on Brinkmann et al. [4] have shown that this order should be at least 38.

Relaxing the cyclic-edge-connectivity conditions, it is easy to find examples of Class 2 Type 2 cubic graphs of cyclic edge-connectivity less than 4. In this paper we presented the first known Type 2 snarks. They are all of girth 4, as well as all known Type 2 cyclically 4-edge-connected cubic graphs distinct from K_4. More generally, it seems that no Type 2 cubic graph with girth greater than 4 is known. Therefore, we also propose the following question.

Question 2. *What is the smallest Type 2 cubic graph with girth at least 5?*

A computer search gave that all cubic graphs with girth 5 and up to 32 vertices are Type 1, so a smallest Type 2 cubic graph with girth at least 5 must have at least 34 vertices.

Question 3. *Is there a girth g so that all cubic graphs with girth at least g are Type 1?*

An astonishing observation we made when evaluating computational results is that among Type 2 cubic graphs there are very few not containing an induced square.

When testing square-free cubic graphs (but allowing triangles), up to 32 vertices (for 32 vertices alone there are 1.988.135.435.965 connected cubic graphs that are square-free) apart from K_4 only two graphs – one with 12 vertices and one with 18 vertices – were found that were Type 2. These graphs are displayed in Figure 8, notice that they have no cycle of length 4 at all. They can be downloaded from the graph database HoG [6] by searching for the keyword `Type2_quad_free`. Allowing squares and forbidding triangles, there are already more than 4.000.000 Type 2 graphs on up to 26 vertices.

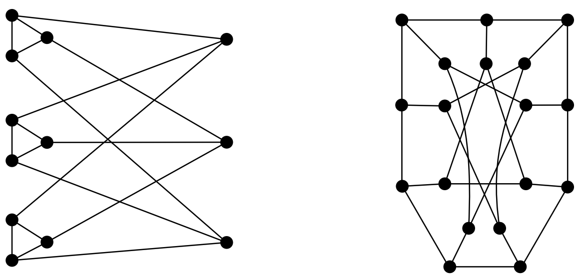

Figure 8: The only two Type 2 square-free cubic graphs of order up to 32. The first one is the graph obtained from $K_{3,3}$ by replacing all vertices of one set of the bipartition by triangles, and the second one is the generalized Petersen graph $P(9,3)$.

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