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A polynomial time algorithm for the single-item lot sizing problem with capacities, minimum order quantities and dynamic time windows

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Abstract

This paper deals with the single-item capacitated lot sizing problem with concave production and storage costs, considering minimum order quantity and dynamic time windows. This problem models a lot sizing where the production lots are constrained in amount and frequency. In this problem, a demand must be satisfied at each period t over a planning horizon of T periods. This demand can be satisfied from the stock or by a production at the same period. When a production is made at period t, the produced quantity must be greater than a minimum order quantity (L) and lower than the production capacity (U). The frequency constraints on the production lots are modeled by dynamic time windows. Between two consecutive production lots, there is at least Q periods and at most R periods. An optimal algorithms in $O(T^9)$ is given. The complexity of the algorithm is reduced to $O(T^7)$ when all the demands are strictly positive.

Keywords: Lot-sizing, polynomial time algorithm, minimum order quantity, capacity constraint, dynamic time window

1 Introduction

This paper deals with a generalization of the single-item capacitated lot sizing problem (CLSP) with fixed capacity. The CLSP consists in satisfying a demand at each time period t over a planning horizon T. The demand is satisfied from stock or by production. Costs incur for each item produced and also when an item is stored between two consecutive periods. A fixed maximum production capacity (U) must be respected. The problem considered in this paper contains a minimum order quantity constraint (MOQ). This constraint imposes that if an item is produced at a given period, the quantity must be greater than or equal to a minimum level L. The U and L values are constant over the T periods. This problem also includes dynamic time windows (DTW). Between two consecutive production lots, there are at least Q periods and at most R periods. These DTW are useful in a long term partnership between two actors, because they allow the decision makers to stabilize the relationship [7]. This problem is noted CLSP-MOQ-DTW in the following.

The single-item capacitated lot sizing problem is known to be NP-Hard [3]. However, some cases are polynomially solvable. This is the case when the capacity is fixed over the T periods. Florian and Klein [5] considered a case where production and holding cost functions are concave. They proposed an exact method with a time complexity in $O(T^4)$. Later van Hoesel and Wagelmans [15] improved the complexity of the algorithm in $O(T^3)$ when the holding costs are linear. A complete survey on the single-item lot sizing problem can be found in [4].

Recently minimum order quantity (MOQ) constraints have been developed. These constraints deal with the production level that must be at least the MOQ if the production is to be started. The CLSP-MOQ has been shown relevant in many industrial contexts, for example Lee [9] has studied an industrial problem where a manufacturer imposes a minimum order quantity to its supplier. Furthermore, Porras and Dekker [13] have worked on an industrial case where the producer imposes minimum order quantities (MOQ) to produce the items. Zhou *et al.* [17] have analysed a class of simple heuristic policies to control stochastic inventory systems with MOQ constraints. They also developed insights into the impact of MOQ constraints on repeatedly ordered items to fit in an industrial context. The first exact polynomial time algorithm was developed by Okhrin and Richter [12]. They solved a special case of the problem in which the unit production cost is constant over the whole horizon and then can be discarded. Furthermore, they assumed that the holding costs are also constant over the T periods, with these restrictions they derived a polynomial time algorithm in $O(T^3)$. Li *et al.* [11] studied the single item lot sizing problem with lower bounds and described a polynomial algorithm in $O(T^7)$ to solve the special case with concave production and storage cost function. Later Hellion *et al.* [6] developed an optimal $O(T^6)$ polynomial time algorithm to solve the CLSP-MOQ with concave costs functions, improving Li *et al.* [11] algorithm. Hellion *et al.* [6] also provide a computational experiment to underline the practical complexity of their algorithm.

The production capacity and MOQ constraints were originally motivated by industrial needs. Considering a retailer ordering from a single supplier, these constraints additionally give the supplier a way to forecast future orders. However, these constraints only affect the quantity of the orders, and both the supplier and the retailer lack temporal informations. To ensure a long-term partnership, actors must guarantee a certain amount of supplied components and regular orders. The time interval between two orders must in a given time window [7].

In the existing literature, time windows have been introduced with several definitions. The delivery time window (also called grace period) was first presented by Lee *et al.* [10]. In their model, each demand d_t must be delivered during a time window. Later, Akbalik and Penz [2] used a similar definition to compare just-in-time and time windows policies (introduced by Brahimi *et al.* [4]). In this problem, items cannot be produced before a defined period. Recently, Absi *et al.* [1] studied two production time window problems, considering lost sales or backlogs. They used dynamic programming to solve their problems. Hwang [8] proposed an $O(T^5)$ algorithm for the production time windows and concave production costs. Van den Heuvel [14] showed that the formulations with production time windows are equivalent to other models: lot sizing with manufacturing options, lot sizing with cumulative capacities and lot sizing with inventory bounds. These two time window definitions were studied by Wolsey [16], he proposed valid inequalities and convex hulls. However, these time window definitions above do not guarantee regular orders. Hellion *et al.* [7] recently presented a new time windows definition in which actors have to agree on a minimum and a maximum number of periods between two orders. Since an order is dependent on the period where the last order occurred, these time windows are dynamic (called DTW as already defined).

In this paper, we extend Hellion *et al.*'s algorithm [6] to the problem with dynamic time windows. The paper is organized as follows: Section 2 describes the problem and introduces the notations. Section 3 presents the necessary definitions and properties to give a polynomial algorithm. However, under specific assumptions the complexity of the algorithm can be slightly reduced: this case is presented in Section 4. Finally, concluding remarks and perspectives are given in Section 5.

2 Problem description and notations

2.1 Description

The single item lot sizing problem consists of satisfying the demands d_t of a product at each period t over T consecutive periods. A demand $d_t \in \mathbb{Z}^+$ may be satisfied by the production of an item at period t (X_t) and/or from inventory (I) available at the end of the period $t-1$ (I_{t-1}). Backlogs are not allowed. The inventory level at the end of a period t is denoted I_t. It is assumed without loss of generality that there is no inventory at the beginning of the first period. The problem is to determine the amount X_t to be produced at each period, satisfying the demands and minimizing the total cost.
The production at each period is constrained by a constant capacity U. The production level is also constrained by the MOQ: L. Each subsequent production level is also constrained by a dynamic time window (DTW). There are at least Q and at most R periods between two consecutive production lots.

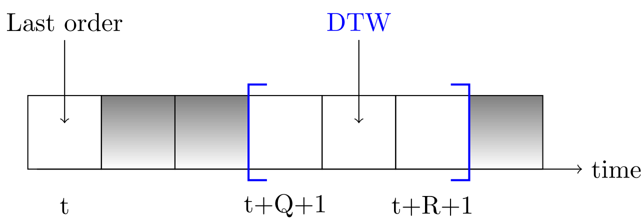

Figure 1: An example of dynamic time window with $Q = 2$ and $R = 4$

Figure 1 illustrates the dynamic time window for $Q = 2$ and $R = 4$. In the example, a lot is produced in period t. Since $Q = 2$, the following lot cannot be produced at neither at period $t + 1$ nor at $t + 2$. Since $R = 4$, at least lot must be produced in the next five

periods. Thereafter, the next lot must be produced between periods $t + 3$ $(t + Q + 1)$ and $t + 5$ $(t + R + 1)$ included. Consequently, in each interval of length 3 at most one lot must be produced. Furthermore, in each interval of length 5, at least one lot must be produced. Note that if $Q = R$ one lot must be produced every $Q + 1$ periods. If $Q = R = 0$, one lot must be produced every period.
The production cost is a concave function of the quantity produced $p_t(X_t)$ and the inventory cost is a concave function of the inventory level $h_t(I_t)$. Note that concave cost functions may include set-up costs.

2.2 Mathematical formulation

The mathematical formulation is now presented. The decision variables are given as follows:

- X_t: quantity of products ordered at period t.
- $Y_t = \begin{cases} 1 & \text{if an order is placed at period } t. \\ 0 & \text{otherwise.} \end{cases}$
- I_t = inventory level at the end of a period t.

The mathematical formulation of the CLSP-MOQ-DTW is then:

$$\text{Min} \sum_{t=1}^{T} p_t(X_t) + \sum_{t=1}^{T} h_t(I_t) \tag{1}$$

$$X_t + I_{t-1} - I_t = d_t \qquad \forall t \in T \tag{2}$$

$$LY_t \leq X_t \leq UY_t \qquad \forall t \in T \tag{3}$$

$$\sum_{t'=t}^{t+R} Y_{t'} \geq 1 \qquad \forall t \in \{1, \cdots, T - R\}, \tag{4}$$

$$\sum_{t'=t}^{t+Q} Y_{t'} \leq 1 \qquad \forall t \in \{1, \cdots, T - Q\} \tag{5}$$

$$X_t, I_t \in \mathbb{R} \qquad \forall t \in T \tag{6}$$

$$Y_t \in \{0, 1\} \qquad \forall t \in T \tag{7}$$

The objective function (1) is to minimize the total cost. Constraint (2) is the flow constraint. Constraint (3) ensures that the maximum capacity and the minimum order quantity are satisfied. The dynamic time windows are given by (4) and (5). Constraints (6) and (7) define the domain of validity the variables.

3 A polynomial time algorithm

Our work is based on the concept of sub-plan introduced by Florian and Klein [5], which leads to a polynomial algorithm. We extend the definitions of Hellion *et al.* [6] to the our problem, *i.e.* including the DTW.

Definition 1. *Regeneration points*
A period t is called a regeneration point if $I_t = 0$.

Definition 2. *Fractional production periods*
A period t is called a fractional production period if $L < X_t < U$.

Definition 3. *Production sequence*
The sequence of production quantities from $u + 1$ to v is noted S_{uv}.

Remark 1. *Let S_{uv} and S_{vw} be two production sequences which are separately feasible. The sequence $S_{uv} \cup S_{vw}$ may be unfeasible, due to the DTW incurred by the last period of production of S_{uv}. When computing S_{uv} we cannot know where the first lot of S_{vw} will be produced. Knowing that $I_v = 0$ is not sufficient because the following demand could be zero. If no assumption on the demand is made, the information given by a production sequence must include the period of the last production. This means that it is possible to know for each feasible S_{uv} where the first lot of S_{vw} can be produced. In order to take this into account, we define a specific production sequence.*

The definition below introduces the DTW-capacity-constrained sequences $S_{ui,vj}$ (a production sequence between the periods u and v). The additional indexes i and j allow the DTW to be respected between two consecutive production sequences. The last lot of the previous production sequence was produced at $u - i$. With R and Q, the DTW is known, and the first lot can be produced. The production of the last lot must take place at $v - j$.

Remark 2. *Let k be the number of null demands at the beginning of the DTW-capacity-constrained sequences. Since $I_u = 0$ the inventory level can be zero at the end of these k periods. In any case $I_t > 0 \; \forall t \in [u + k + 1, \cdots, v - 1]$.*

Definition 4. *DTW-capacity-constrained sequences*
$S_{ui,vj}$ is a DTW-capacity-constrained sequence if the following conditions are verified:

- *u and v are regeneration points i.e. $I_u = I_v = 0$;*
- *The demand d_t for $t = \{u + 1, \cdots, v\}$ is satisfied;*
- *$X_t \geq L$ for at least one t in $[u - i + Q + 1, \cdots, u - i + R + 1]$*
- *$X_{v-j} \geq L$ and $X_t = 0$ for $t \in [v - j + 1, \cdots, v]$*
- *The production X_t for $t \in \{u + 1, \cdots, v\}$ is either equal to 0, L or U, except for at most one period which can be a fractional production period.*
- *$S_{ui,vj}$ respects the DTW constraints.*

- $I_t = 0$ *before the first production period and* $I_t > 0$ *after the first production period.*

The dominance of the DTW-capacity-constrained sequences is proved later.

Property 1. *Given two* DTW*-capacity-constrained sequences* $S_{ui,vj}$ *and* $S_{wk,zl}$*, the succession of these production sequences is feasible if and only if* $v = w$ *and* $j = k$*.* $S_{ui,vj}$ *and* $S_{wk,zl}$ *are then said compatible.*

Proof. Since a DTW-capacity-constrained sequence begins where the previous one ends, $v = w$. As stated above, the last lot of $S_{ui,vj}$ is produced at the period $v - j$. Index wk means that the last lot of the previous sequence was produced at $w - k$. As $v = w$, the DTW can be respected if $j = k$. □

Due to the MOQ constraint, the final storage could be strictly positive. At this time, we consider that $I_T = 0$. The case for which $I_T > 0$ will be considered at the end of this section, in the Property 5.

Property 2. *A solution to the CLSP-MOQ-DTW problem can be seen as succession of compatible* DTW*-capacity-constrained sequence.*

Proof. Assuming that $I_k = 0$ for some $k \in \{1...n-1\}$. An optimal solution of CLSP-MOQ-DTW can be found by independently finding solutions to the problems for the first k periods and for the last $T - k$ periods. However the DTW have to be respected, inside and between the production sequences. Consequently, a production plan can be seen as a succession of compatible DTW-capacity-constrained sequences, in accordance to Property 1. □

Let us build a directed acyclic graph ($\mathcal{G}$) as follows. Define $T + 1$ vertices labeled from 0 to T. All these vertices are duplicated on $R + 1$ level, and each vertex is labeled (t, i) referring to its period $t \in 0 \dots T$, and its level $i \in 0 \dots R$, respectively. A vertex (t, i) signifies that the period t is a *regeneration point*. For convenience two additional vertices *start* and *end* are added. Each arc $(u, i) \to (v, j)$ models a DTW-capacity-constrained sequence $S_{ui,vj}$. A number of arcs can be immediately discarded, since the production at periods $u - i$ and $v - j$ is unfeasible considering a given DTW. The vertices *start* and *end* are connected at a null cost with all the nodes $(0, i)$ and (T, i), $\forall i \in [0, \cdots, R]$, respectively. For each pair $\{(t, i),(T, j)\}$, we have to add two arcs to the graph, one representing $I_T = 0$ (Property 4 is given below), and one representing $I_T \neq 0$ (Property 5 is given at the end of the section). Figure 2 shows an instance of a ($\mathcal{G}$) considering $T = 3$, $Q = 1$ and $R = 1$. A shortest path between the nodes *start* and *end* leads to an optimal production plan.

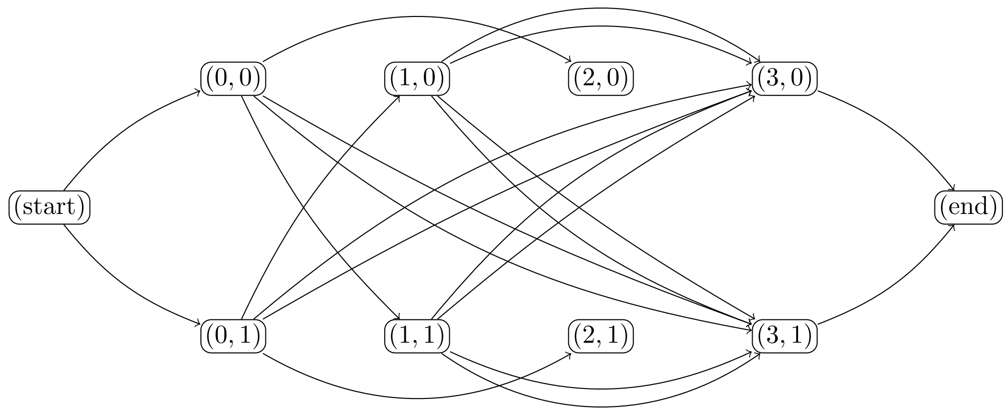

Figure 2: An example of graph $\mathcal{G}$ considering $T = 3$, $Q = 1$ and $R = 1$

Property 3. *Let us consider an interval of periods $[u, v]$ such that $I_u = I_v = 0$. DTW-capacity-constrained sequences are dominant for CLSP-MOQ-DTW (i.e. at least one optimal solution is a DTW-capacity-constrained sequence).*

Proof. To prove this result, we show that if a production sequence $S_{ui,vj}$ is not a *DTW*-capacity-constrained sequence, it cannot be an extreme point of the polyhedron defined by constraints 2 to 7, and consequently is dominated by an other solution. In order to prove this result, we show that the solution $S_{ui,vj}$ is a convex combination of two other feasible solutions.
Let us consider a solution $S_{ui,vj}$ such that both u and v are regeneration points. $I_u = I_v = 0$, $I_t \neq 0$ for $t \in \{t', \cdots, v-1\}$ where t' is the first lot produced in a sequence (see Remark 2) in such a way that there exist at least two *fractional production periods a* and b such that $t' \leq a < b \leq v$ and $L < X_a, X_b < U$. Since $S_{ui,vj}$ is a feasible sequence for this problem, the *DTW* are respected. Consequently, we can relocate a small amount of production between X_a and X_b as follows. Let us define ω as the biggest production quantity we can relocate keeping the solution feasible, and without changing other production levels. Then:

$$\omega = \min\{U - X_a \,;\, U - X_b \,;\, X_a - L \,;\, X_b - L \,;\, \min_{t=a}^{b-1} I_t\}$$

By relocating $\frac{1}{2}\omega$ from a to b, we obtain a solution $S'_{ui,vj}$. The production plan $S'_{ui,vj}$ is obviously feasible and the *DTW* constraints still hold. Symmetrically, by relocating $\frac{1}{2}\omega$ from b to a, we obtain a valid solution $S''_{ui,vj}$. However, $S_{ui,vj} = \frac{1}{2}S'_{ui,vj} + \frac{1}{2}S''_{ui,vj}$, proving that $S_{ui,vj}$ is not an extreme point. Therefore $S_{ui,vj}$ is not the unique optimal solution, and it is dominated. □

Property 4. *A DTW-capacity-constrained sequence $S_{ui,vj}$ can be computed in $O(T^5)$.*

Proof. Let us define α (resp. β) as the number of periods in which the production is equal to U (resp. L). The fractional production is noted ε with $L < \varepsilon < U$. Using $D_{uv} = \sum_{t=u+1}^{v} d_t$, the total demand for the sequence, we can write:

$$\alpha U + \beta L + \varepsilon = D_{uv} \tag{8}$$

In some cases, $\alpha U + \beta L = D_{uv}$. This means that there is no fractional production period in the DTW-capacity-constrained sequence. In this case we note $\varepsilon = 0$. Note also that Okhrin and Richter [12] define the same parameters, called k and K in their paper.

First of all, we will prove that the number of triplets $(\alpha, \beta, \varepsilon)$ is in $O(T)$. This will be followed by the proof that it is possible to compute a DTW-capacity-constrained sequence in $O(T^5)$.

Consider a given β, we have only one feasible value for α, which is $\lfloor \frac{D_{uv} - \beta L}{U} \rfloor$. The magnitude of β is in $O(T)$, thus the number K of triplets $(\alpha, \beta, \varepsilon)$ is in $O(T)$. At most there are K different ε, and they are noted ε_k for $\{k = 1, \cdots, K\}$.

We must now prove that the best production plan can be found in polynomial time.

Considering a single DTW-capacity-constrained sequence $S_{ui,vj}$, another directed acyclic graph ($\mathcal{UV}$) can be built (see Figure 3). For each period t such as $u + 1 \leq t \leq v$ we have a node for each feasible cumulative production level. We have an arc labeled x between 2 nodes a and b only if a production $X_b = x$ is feasible, with x being either L, U or ε_k. Each arc is weighted with the associated cost (production and storage). A shortest path allows the minimization of the total cost of this DTW-capacity-constrained sequence.

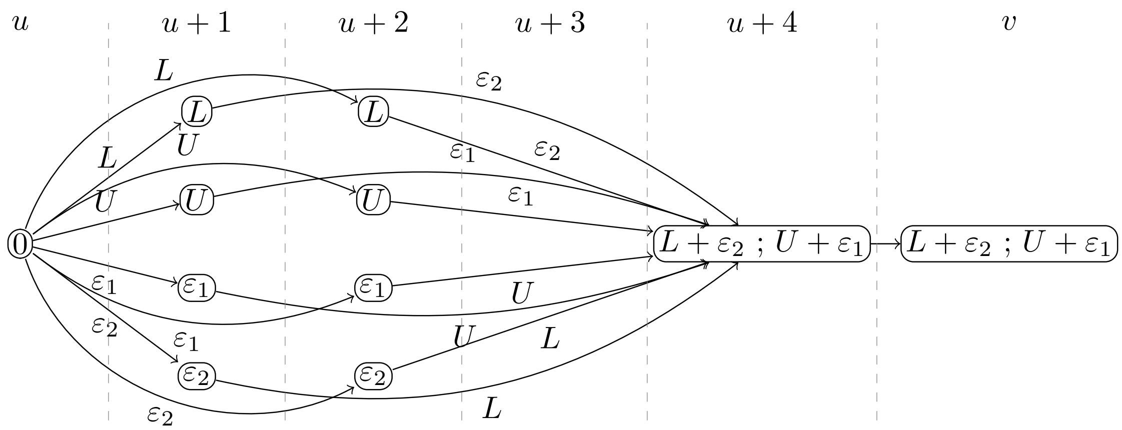

Figure 3: $S_{u1,v1}$: an example of graph $\mathcal{UV}$ with 5 periods, $Q = 1$ and $R = 2$

The Figure 3 shows how a given DTW-capacity-constrained sequence can be computed. We choose to display $S_{u1,v1}$, considering 5 periods and $Q = 1$ and $R = 2$. In this example, we have to choose between producing at $u + 1$ or $u + 2$, knowing that the last production was at $u - 1$. Lots must also be produced at $u + 4$, because $u + 4 = v - 1$.

Considering a given triplet k: $(\alpha, \beta, \varepsilon_k)$, the number of nodes at each level of the graph

is in $O(T^2)$, since the magnitude of α and β are both in $O(T)$. Therefore for this triplet we have $O(T^3)$ nodes. Each node has at most 3 predecessors on each level, leading to $O(T)$ predecessors. The evaluation of a node for the triplet k can be made in $O(T^4)$. The number of triplets is in $O(T)$ and consequently the time complexity for finding an optimal solution is in $O(T^5)$.
This concludes the proof of property 4. □

Unfortunately, as mentioned above, due to the MOQ constraint an optimal solution can have items in stock at T. Let us assume that the storage cost $h_{T-1}(I_{T-1})$ for a positive value of I_{T-1} is very high. If the production cost at period T is low and if the demand at period T respects $0 < d_T < L$, the best strategy could be to produce L at the last period leading to a storage of $I_T = L - d_T$ items. Consequently, we must study the sequences $\widehat{S}_{ui,Tj}$ where u is a *regeneration point* and such that $I_T \neq 0$.

Property 5. *An optimal production sequence $\widehat{S}_{ui,Tj}$ such that u is a regeneration point, and $I_T \neq 0$ can be computed in polynomial time.*

Proof. First of all, $\widehat{S}_{ui,Tj}$ cannot contain a fractional production period; if $\widehat{S}_{ui,Tj}$ contains a fractional production period at a period t' of value ε, it is possible to decrease the production at this period by $min\{\varepsilon - L, \min_{t=t'}^{T} I_t\}$. Furthermore, if $\widehat{S}_{ui,Tj}$ contains a period where the production is maximum (U) at a period t', we can easily decrease the production because the storage levels are strictly positive from t' to T. Then $\widehat{S}_{ui,Tj}$ cannot contain a production level at U.

Consequently, the sequence $\widehat{S}_{ui,Tj}$ only has production periods at L and 0. Furthermore, $I_T < L$, otherwise we can suppress one of the productions, and then there is only one value for β which is:

$$\beta = \lceil \frac{D_{u+1T}}{L} \rceil$$

Thus there is only one feasible triplet $(\alpha, \beta, \varepsilon)$ in this production sequence. The best production plan can now be computed in $O(T^4)$ (see Property 4) with a graph similar to the one presented in the Figure 3. □

We can now derive a polynomial time algorithm (called *DTW-HMP*) from the previous properties.

Theorem 1. *Algorithm HMP-DTW gives an optimal solution for the CLSP-MOQ-DTW problem with concave production and storage costs in time $O(T^9)$*

Proof. An optimal solution is given by a succession of sub-sequences between two regeneration points (Property 2). One of the shortest paths in graph $\mathcal{G}$ is an optimal solution (see Figure 2). The construction of the graph $\mathcal{G}$ is in $O(T^9)$. Furthermore, we have $O(T^4)$ arcs in graph $\mathcal{G}$ and each arc can be computed in $O(T^5)$ (see Properties 4 and 5), leading to a time complexity of $O(T^9)$. Finding one of the shortest paths in $\mathcal{G}$ can be made in $O(T^4)$. We therefore conclude that the time complexity of the algorithm *HMP-DTW* is in $O(T^9)$ (Algorithm 1). □

Algorithm 1 DTW-HMP

Build the graph ($\mathcal{G}$)
for each edge E of ($\mathcal{G}$) **do**
  Build the corresponding graph ($\mathcal{UV}$)
  Perform a shortest path algorithm to find an optimal production sequence and its cost
  Set the weight of the edge E with the obtained cost
end for
Perform a shortest path algorithm on graph ($\mathcal{G}$) to find the best production sequence

4 An $O(T^7)$ algorithm when the demand are strictly positive

In this section it is shown that, with the assumption that $d_t > 0\ \forall t \in T$, the complexity of the algorithm can be slightly reduced. Let us called this new problem CLSP-MOQ-DTW with positive demands (CLSP-MOQ-DTW-PD). Consider again two production sequences S_{uv} and S_{vw} such that u, v and w are regeneration points. S_{uv} and S_{vw} are separately feasible, that is to say they both respect all the constraints, including the DTW. The production sequence is defined as $s = S_{uv} \cup S_{vw}$.

Remark 3. *Consider that $I_v = 0$ and $d_{v+1} > 0$. When computing S_{vw} a lot must be produced at $v+1$ to satisfy the demand d_{v+1}. This means that each computed S_{uv} has to allow the production at $v+1$ for the following production sequence. The information about the last period of production of S_{uv} can therefore be discarded. This decreases the number of DTW-capacity-constrained sequence to compute.*

This in turn, leads to another extension of the original definition.

Definition 5. *PD-capacity-constrained sequences*
S_{uv} is a PD-capacity-constrained sequence if the following conditions are verified:

- *u and v are regeneration points i.e. $I_u = I_v = 0$;*
- *The demand d_t for $t = \{u+1, \cdots, v\}$ is satisfied;*
- *For all $t \in \{u+1, \cdots, v-1\}$, $I_t > 0$ i.e. t is not a regeneration point;*
- *The production X_t for $t \in \{u+1, \cdots, v\}$ is equal to 0, U or L, except for at most one period which can be a fractional production period.*
- *S_{uv} respects the DTW constraints, considering that v+1 is a production period.*

We can easily extend the DTW-capacity-constrained sequences properties to the PD-capacity-constrained sequences: two PD-capacity-constrained sequences can follow each other, a succession of them can solve the problem and they are dominant (Properties 1, 2 and 3, respectively). However, as said above, the number of sequences to compute has decreased. The general graph implied (say graph $\mathcal{G_{PD}}$) is a simplification of the graph $\mathcal{G}$ (Figure 2). The only difference is that the $T+1$ vertices no longer need to be duplicated on $R+1$ levels.

Property 6. *A PD-capacity-constrained sequence can be computed in $O(T^5)$.*

Proof. Considering a single PD-capacity-constrained sequence $S_{u,v}$, a directed acyclic graph ($\mathcal{UV}_{\mathcal{PD}}$) can be built (see Figure 4 for an example). It is similar to the graph ($\mathcal{UV}$) presented in Figure 3.

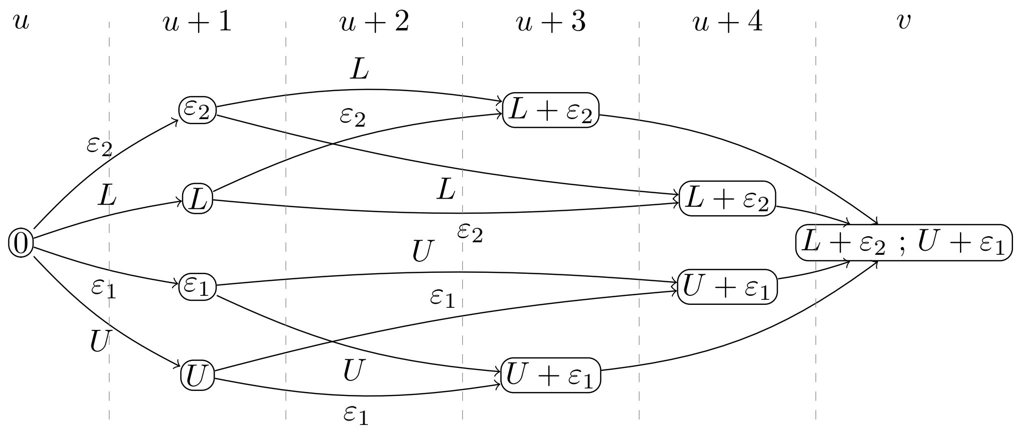

Figure 4: An example of graph $\mathcal{UV}_{\mathcal{PD}}$ with 5 periods, $Q = 1$ and $R = 2$

The Figure 4 shows how a PD-capacity-constrained sequence S_{uv} can be computed, considering 5 periods and $Q = 1$ and $R = 2$. In the example shown, lots must be produced at $u + 1$. The period $v + 1$ must also be allowed to be in a DTW. Thus there is the choice between producing at $u + 3$ or $u + 4$.
As in Property 4 the time complexity for finding an optimal solution is in $O(T^5)$. □

Theorem 2. *Assuming that $d_t > 0$ ($\forall t \in T$), the algorithm HMP-DTW-PD gives an optimal solution for the CLSP-MOQ-DTW-PD with concave production and storage costs in time in $O(T^7)$*

Proof. The proof here is the same as the one presented in Theorem 1. The only difference is that we only have $O(T^2)$ arcs in graph ($\mathcal{G}_{\mathcal{PD}}$), instead of the initial $O(T^4)$ arcs in graph $\mathcal{G}$. This leads to the new time complexity of $O(T^7)$. □

5 Conclusion

The focus of this paper is a generalization of the capacitated single item lot sizing problem. In this paper, the production levels are bounded by a minimum order quantity and a maximum capacity when a lot is produced. Furthermore, the frequency of the produced lots is modeled by dynamic time windows, recently introduced by Hellion [7]. Between two consecutive production lots, there is are least Q periods and at most R periods. Both production and storage cost functions are concave. We proposed an $O(T^9)$ exact algorithm that generalizes Hellion and al's [6] algorithm. A less complex algorithm in $O(T^7)$

is also provided in a case where all the demands are strictly positive. The theoretical complexity of Algorithm HMP-DTW appears difficult to improve with general concave costs. In future work, it would be interesting to determine if this theoretical complexity can be decreased when the cost structure is limited to a fixed cost plus a linear cost. Other assumptions such as the consideration of only linear storage costs [12] may lead to other improvements in the complexity of the algorithm.

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