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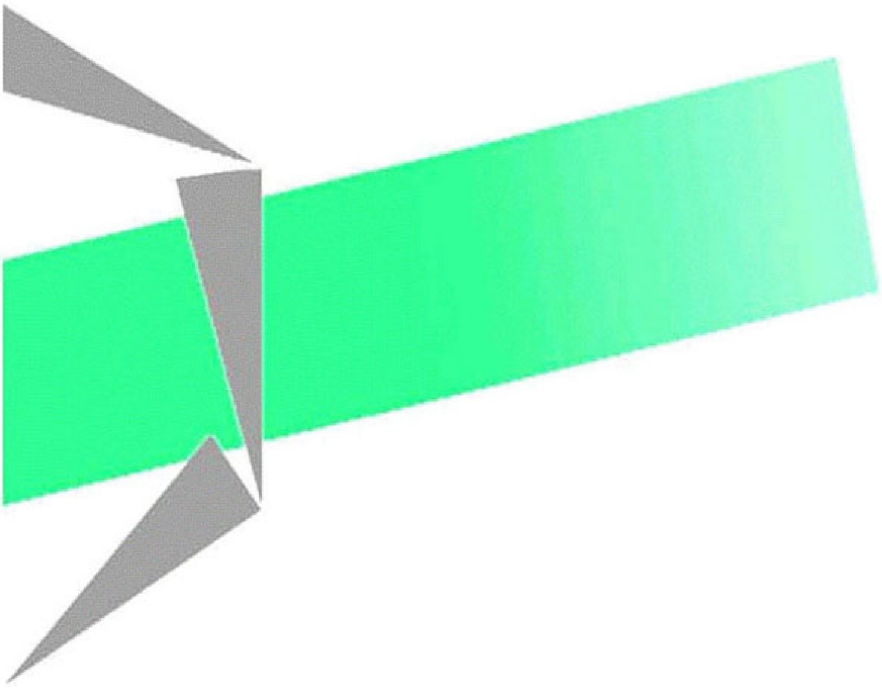

Additions to Lawler's min-max cost algorithm: Optimality conditions and uncertainty

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Abstract

The well-known $O(n^2)$ min-max cost algorithm of Lawler [2] was developed to minimize the maximum cost of jobs processed by a single machine under precedence constraints. We propose two results related to Lawler's algorithm. The first one concerns necessary and sufficient conditions for the sequence optimality when all the cost functions are strictly increasing. The second result concerns the scheduling on a single machine under both uncertainty of some job's parameters and precedence constraints. In the latter problem, each job has an associated non-decreasing cost function of a special (decomposable) form. The cost function for each job depends on the job completion time and on an additional parameter value. The aim is to find a schedule for processing the jobs that is feasible with respect to the given precedence constraints and that minimizes the maximum cost over all the jobs. The class of the functions under consideration includes known functions such as the maximum lateness and the maximum delivery time. For the problem under uncertainty (where for each job, we know only an interval for possible values of the additional parameter), we derive an $O(n^2)$ algorithm for constructing a schedule that minimizes the maximal regret criterion. To obtain this schedule, we use Lawler's algorithm as a part of our technique.

1 Introduction

Lawler [2] considered the following single machine scheduling problem. Each of n simultaneously available jobs is to be processed without interruption on a single machine, which can process at most one job at a time. Job $j,\ j = 1, ..., n$, has a processing time $p_j \geq 0$. To each of the jobs j there is assigned a non-decreasing cost function $\Phi_j(t)$ that specifies the cost incurred if job j is completed at time t. There are precedence constraints represented by an acyclic directed graph $G = (V,\ E)$ with n vertices, where each vertex $v \in V$ corresponds to one of the jobs. Note that graph G contains all

transitive arcs. If job j_1 precedes job j_2 ($j_1 \rightarrow j_2$), i.e., $(j_1,\ j_2) \in E$, then the processing of job j_2 cannot start before the completion of processing job j_1. The objective is to find a feasible schedule s that minimizes the maximum cost $\Phi_{\max}(s) = max_{j=1,...n}\Phi_j(C_j(s))$, where $C_j(s)$ is the completion time of job j under schedule s.

A feasible schedule may be presented by the sequence $\pi = (\pi(1), ..., \pi(n))$ of jobs. Here $\pi(k)$ is the job in position k of permutation π. Denote by $C_j(\pi) = p_{\pi(1)} + \ldots + p_{\pi(k)}$ the completion time of job $j = \pi(k)$ allocated in position k of π. A feasible permutation that minimizes $\Phi_{\max}(\pi)$ is called *optimal*. The formulated problem $1|prec|\Phi_{\max}$ is solved by Lawler's min-max cost algorithm that may be described as follows:

Algorithm 1

1. *Set $k = n$ and $T = \sum_{j=1}^{n} p_j$.*
2. *Let Ω be the set of terminal vertices of graph $G = (V, E)$, i.e., vertices without successors.*
3. *Find vertex $u \in \Omega$ such that $\Phi_u(T) = \min\{\Phi_v(T)|v \in \Omega\}$ (brake ties arbitrarily).*
4. *Set $\pi(k) = u$, $k = k - 1$, $T = T - p_u$ and modify graph G by setting $V = V \setminus \{u\}$.*
5. *If $k > 0$, then go to Step 2. If $k = 0$, then the optimal permutation is constructed.*

The time complexity of the algorithm does not exceed $O(n^2)$. It is supposed that graph G is represented by its adjacency matrix and we assume that each evaluation of each function $\Phi_j(t)$ can be done in $O(1)$ time.

Lawler's algorithm provides a single very particular optimal solution whereas there may exist a group of such solutions. Lin and Wang [3] derived necessary and sufficient conditions for the optimality of a schedule for a number of scheduling problems without precedence constraints (for problem $1||L_{\max}$, in particular; here $L_{\max}$ is the maximum lateness). In our paper, we present necessary and sufficient conditions for the optimality of a schedule for the general problem $1\,|\,prec\,|\,\Phi_{\max}$ with strictly increasing cost functions (for the case of non-decreasing cost functions, the conditions are sufficient).

The traditional scheduling models, in which values of all parameters are known numbers, fail in many practical applications. For a number of the real-world situations some parameters (processing and transportation times, release and due dates etc) are not precisely known in advance. Everything

that is known is the set of possible values for each of the parameters. These parameters are not controllable, each of them takes any values from the given set, and the choice of the particular value does not depend on the will of the decision maker. Such parameters are called uncertain, the model and a correspondent optimization problem are the model and the problem under uncertainty.

For a problem under uncertainty, each possible choice of the parameter values generates a scenario. Given a scenario, we have a deterministic problem that may be solved using this or that particular method. The main peculiar feature of any problem under uncertainty is that we do not know a priory, which particular scenario will be realized.

One of the most popular approaches of dealing with problems under uncertainty is the worst-case or robust approach [5]. Under this approach, the decision maker expects that the real scenario will be the worst possible from the point of view of the problem to be solved, and constructs a solution that is the best one under the expected worst scenario. Intuitively, a robust solution is a solution that remains suitable whatever scenario finally occurs. Within the worst-case approach, there are some different concepts, which scenario is the worst. The two main concepts are the worst objective function value and the maximum deviation of the objective function value from the optimum.

To simplify the description, suppose that we are dealing with a scheduling problem and the aim is to minimize an objective function. Then in the first concept, the decision maker's aim is to minimize the objective function under maximally unsuitable values of the uncertain parameters. In view of the later concept, the decision maker tries to find a schedule that minimizes the maximum possible difference between the objective function values for the schedule obtained and the optimal schedule for a real scenario. Respectively, the approaches are called minmax and minmax regret approaches [5].

To be more formal, denote the set of all possible scenarios by X and the set of all feasible schedules by S. Denote the objective function by $F(x, s)$ (here we are pointing out that the objective function depends on schedule $s \in S$ and on the scenario $x \in X$). Then the minmax approach is to minimize the function

$$\max\{F(x, s)|x \in X\},$$

whereas the minmax regret approach is to minimize the function

$$\max\{F(x, s) - F(x, s^*(x))|x \in X\},$$

where $s^*(x)$ is an optimal schedule for the scenario x.

It seems, that Wald [7] was the first who proposed the minmax approach to deal with uncertainty and Savage [6] is the author of the minmax regret approach.

We consider a single machine sequencing problem with the precedence constraints being a special case of the above problem of minimizing the maximum cost considered by Lawler [2]. In our problem, the cost functions of jobs are not arbitrary non-decreasing functions (as in [2]), but so-called decomposable non-decreasing functions. Each job j has two parameters that are the processing time p_j and an additional parameter λ_j. We study the problem supposing that the additional parameter is uncertain and the set of possible values of the additional parameter for each job is an interval. In our case, the set of all possible scenarios is the Cartesian product of these intervals. We examine the two mentioned approaches: the minmax approach and the minmax regret approach. In both cases we actively use Lawler's algorithm.

The only known result for the problems of the mentioned type is presented by Kasperski in [4], where the problem $1|prec|L_{\max}$ was considered under uncertainty of processing times and due dates of the jobs (the set of possible values for each of the uncertain parameters is an interval). In [4], a $O(n^4)$ algorithm has been developed for constructing a schedule that is optimal with respect to the minmax regret criterion. In our model, we consider a more general objective function that includes $L_{\max}$ as a special case. At the same time, we are dealing with a single uncertain parameter. One of our results is an $O(n^2)$ algorithm for constructing a schedule that is optimal with respect to the minmax regret criterion.

We denote by $A_G(v)$ and $B_G(v)$ the set of all successors of vertex v in graph G and the set of all predecessors of v, respectively.

This paper is organized as follows: In Section 2 we consider a deterministic version of the main problem and formulate necessary and sufficient conditions for a feasible permutation to be optimal, besides, we present some related results that are used in the subsequent sections. In Section 3 we are dealing with the main problem under uncertainty. We present $O(n^2)$ algorithms for constructing schedules that are optimal with respect to the minmax criterion (Subsection 3.1) and to the minmax regret criterion (Subsection 3.2). In the last section, we summarize our results and outline possible directions for the further research.

2 Conditions of optimality

For our problem $1|prec|\Phi_{\max}$, we require first that all functions $\Phi_j(t)$ are supposed to be strictly increasing. Compared to [3] (where necessary and sufficient conditions of a permutation optimality were derived for problem $1||L_{\max}$), difficulties arise in connection with the precedence constraints and with the presence of zero-length jobs.

Let a feasible permutation π be given. We describe a job exchange, as Lawler did in his original paper, but in a slightly modified form. Let us represent the job sequence π schematically as follows:

π :

A	u	B	v	C

with segments A, B and C of sequence π and jobs v and u with $p_u > 0$ and $u \not\to v$ (i.e., $u \notin B_G(v)$) if there is any. We consider only jobs u that are close to v so that the set W of all successor jobs of u in B contains only zero-length jobs. Set $U = (u, W)$ and move subpermutation U to the right just following v. Then we obtain the feasible permutation π':

π' :

A	B'	v	u	W	C

where $B' = B \setminus W$. Here it is understood that the inner order of jobs in B' and W is saved from the order in π. The following definition describes the transitions from π to π' that yield some kind of strict improvement, at least locally.

Definition 1 *Given a feasible permutation π and a job v in π with completion time t. We say that job v has a local improvement in π if there exists job u, as described above, which satisfies the conditions $\Phi_j(t) < \Phi_v(t)$ for all $j \in \{u\} \cup W$.*

The effect of a local improvement of v is immediate. For π' we get the following properties:

1. The positions of jobs $j \in A \cup C$ are unchanged. Hence these jobs keep the same completion times and costs.
2. Let R be the set of all remaining jobs, then the maximum cost of the jobs $j \in R$ is strictly reduced, i.e., $\max_{j \in R} \Phi_j(C_j(\pi')) < \max_{j \in R} \Phi_j(C_j(\pi))$.

The validity of the property follows from the conditions $p_u > 0$ and $\Phi_j(t) < \Phi_v(t)$ for all $j \in \{u\} \cup W$.

3. We have $\Phi_{\max}(\pi') \leq \Phi_{\max}(\pi)$.

Definition 2 *Job v is called critical in permutation π, if $\Phi_{\max}(\pi) = \Phi_v(C_v(\pi))$.*

Theorem 1 *If all functions $\Phi_j(t)$ are strictly increasing, then a feasible permutation π is optimal for problem $1 \,|\, prec \,|\, \Phi_{\max}$ if and only if there is a critical job in π without any local improvement.*

Proof. *Necessity.* Let π be an optimal permutation. We prove the necessity by induction on the number of critical jobs.

Consider a situation with the single critical job v. If job v has a local improvement in π, then in view of the above properties of the local improvement there exists a feasible permutation π' such that $\Phi_v(C_v(\pi')) < \Phi_v(C_v(\pi))$ and the values of the cost functions of all other jobs are less than $\Phi_v(C_v(\pi))$. So, $\Phi_{\max}(\pi') < \Phi_{\max}(\pi)$ that contradicts the optimality of π.

Suppose that the theorem conditions are necessary for the optimality of any permutation with less than $l \geq 2$ critical jobs.

Consider the case with l critical jobs. If each of the critical jobs has a local improvement, then consider the critical job $v = \pi(k)$ in the left-most position. Acting similarly to the case with the single critical job we can move job u (see the definition of the local improvement) along with its successors in positions j, $j < k$, to the positions to the right of v. In the result, we obtain a feasible permutation π', in which job v is not a critical job and the general number of critical jobs in π' equals $l - 1$. Besides, $\Phi_{\max}(\pi') \leq \Phi_{\max}(\pi)$, i.e., π' is an optimal permutation. All these $l - 1$ critical jobs are in the same positions as in permutation π. Since for π, each of these $l - 1$ critical jobs has a local improvement, each of them has a local improvement in permutation π' as well. It means that for the optimal permutation π' with $l - 1$ critical jobs the theorem conditions are violated. That contradicts our induction hypothesis.

Sufficiency. Suppose that in a feasible permutation π, there exists critical job $v = \pi(k)$ without local improvements. We have $\Phi_{\max}(\pi) = \Phi_v(C_v(\pi)) \geq \Phi_j(C_j(\pi))$ for all $j \in \{1, \ldots, n\}$. Let π' be an optimal permutation. If $C_v(\pi') \geq C_v(\pi)$, then $\Phi_{\max}(\pi') \geq \Phi_{\max}(\pi)$ and π is an optimal permutation.

Denote $L(v) = \{\pi(l) \notin B_G(v) |\ l < k,\ p_{\pi(l)} > 0\}$ (remind that job v is in position k in π).

Suppose that for an optimal permutation π' we have $C_v(\pi') < C_v(\pi)$. Then there exists at least one job $\pi(l) \in L(v)$ on the right of v in π' (otherwise we would have $C_v(\pi') \geq C_v(\pi)$). Let $\pi(l)$ be the right-most job from $L(v)$ in π', then $C_{\pi(l)}(\pi') \geq C_v(\pi)$. Since v does not have a local improvement in π, we have $\Phi_{\pi(l)}(C_v(\pi)) \geq \Phi_v(C_v(\pi))$. Therefore, $\Phi_{\pi(l)}(C_{\pi(l)}(\pi')) \geq \Phi_{\pi(l)}(C_v(\pi)) \geq \Phi_v(C_v(\pi))$ and $\Phi_{\max}(\pi') \geq \Phi_{\max}(\pi)$, i.e., π is again an optimal permutation. ∎

Corollary 1 *If all $\Phi_j(t)$ are non-decreasing functions, then a feasible permutation π is optimal for problem $1\,|\,prec\,|\,\Phi_{\max}$ if there is a critical job in π without any local improvement.*

Definition 3 *Given a feasible permutation π and a job v in π with completion time t. Let job u be on the left of v in π and U is the set of jobs between u and v in π. We say that job v has a weak improvement in π if $\Phi_u(t) < \Phi_v(t)$ and none of the jobs in $U \cup \{v\}$ is a successor of u.*

Lemma 1 *If all functions $\Phi_j(t)$ are non-decreasing and job v has a weak improvement in a feasible permutation π, then there exists a feasible permutation π' such that $C_v(\pi') \leq C_v(\pi)$, $\Phi_u(C_u(\pi')) < \Phi_v(C_v(\pi))$ and $\Phi_{\max}(\pi') \leq \Phi_{\max}(\pi)$.*

Proof. To construct the mentioned permutation π', it is enough to move job u from its position to position of job v in π, simultaneously shifting all jobs of the set $U \cup \{v\}$ one position to the left. ∎

Evidently, if a job has no weak improvement in a feasible permutation π, then it also has no local improvement in π.

Corollary 2 *If all functions $\Phi_j(t)$ are non-decreasing, then a feasible permutation π is optimal for problem $1\,|\,prec\,|\,\Phi_{\max}$ if there is a critical job in π without any weak improvement.*

Corollary 3 *If permutation π for problem $1\,|\,prec\,|\,\Phi_{\max}$ is constructed by Algorithm 1, then none of the jobs has a weak improvement in π.*

Proof. Suppose that job $\pi(k)$ has a weak improvement in π. Consider the $k-$th run of Step 3 in Algorithm 1. The definition of the weak improvement means that in this run, there is at least one job j in the current set Ω such that $\Phi_j(T) < \Phi_{\pi(k)}(T)$. In accordance with the description of Step 3, job j had to be chosen to be placed in position k in π instead of job $\pi(k)$. ∎

3 Decomposable cost functions with uncertainty

Now, we formulate our main problem under uncertainty. We restrict the cost functions of jobs to be decomposable into a sum of two components. The first component is a monotone function that depends on the job completion time. The second component is an uncertain parameter.

Each of n simultaneously available jobs is to be processed without interruption on a single machine, which can process at most one job at a time. Job j has a processing time $p_j \geq 0$ and an additional parameter λ_j that takes rational values. Each job j has an associated cost function $\Phi_j(t, \lambda_j) = \varphi(t) + \lambda_j$ that presents a cost to be incurred if the job processing is completed at time t. Here $\varphi(t)$ is a non-decreasing function. There are precedence constraints represented by a directed graph $G = (V,\ E)$ with n vertices. The aim is to find a job permutation π that is feasible with respect to the precedence constraints and that minimizes the maximum cost

$$\Phi_{\max}(\pi) = \max\{\Phi_j(C_j(\pi), \lambda_j) | j = 1, \ldots, n\}.$$

The best known examples of functions $\Phi_j(t, \lambda_j) = \varphi(t) + \lambda_j$ are

– lateness of job $j : L_j(t,\ d_j) = t - d_j$, where $d_j \geq 0$ is a due date of job j; here $\lambda_j = -d_j$, and
– delivery time of job $j : D_j(t,\ q_j) = t + q_j$, where $q_j \geq 0$ is the transportation time of job j (after completing the processing, job j is to be delivered to its end user and the delivering takes q_j time units); here $\lambda_j = q_j$.

We consider the problem above under uncertainty of the additional parameter λ_j: for each job j there are given the lower bound λ_j^- and the upper bound λ_j^+ of possible values of λ_j. The actual value of λ_j is unknown, it may take any value from the interval $[\lambda_j^-, \lambda_j^+]$ regardless of the will of a decision maker.

We denote the uncertain version of the problem by

$$1|prec;\ \lambda_j \in [\lambda_j^-,\ \lambda_j^+]|\Phi_{\max}.$$

Further, we suppose that the calculation of each value of function $\varphi(t)$ takes a constant time.

Suppose we are given particular values λ_j^-, λ_j^+ and p_j for each $j \in \{1, ..., n\}$, a particular function $\varphi(t)$ and a particular graph G presenting the precedence constraints. If we choose the values of the uncertain parameters in any feasible way, i.e., $\lambda_j \in [\lambda_j^-, \lambda_j^+]$ for each $j \in \{1, ..., n\}$, we get a *scenario* I for the problem $1|prec;\ \lambda_j \in [\lambda_j^-,\ \lambda_j^+]|\Phi_{\max}$ under uncertainty.

We denote by $\mathcal{I}$ the set of all scenarios. Clearly, set $\mathcal{I}$ is infinite if $\lambda_j^- \neq \lambda_j^+$ for at least one $j \in \{1, ..., n\}$.

Given a scenario $I \in \mathcal{I}$, we have a deterministic version of the problem in question and the only remaining problem is to find an optimal permutation (that may be done using Lawler's algorithm).

For a given scenario $I \in \mathcal{I}$ and a permutation π, we write $\Phi_{\max}(I, \pi)$ to denote the corresponding value of the objective function and we write $\Phi^*_{\max}(I)$ to denote the minimum value of the objective function for the scenario I. We denote an optimal permutation for the scenario I by $\pi^*(I)$.

Below we consider two robust approaches to solve the problem $1|prec;\ \lambda_j \in [\lambda_j^-,\ \lambda_j^+]|\Phi_{\max}$ under uncertainty: minmax and minmax regret approaches.

3.1 Minmax criterion

Since each function $\Phi_j(t, \lambda_j) = \varphi(t) + \lambda_j$ is strictly increasing on the parameter λ_j, for any permutation π the value of the objective function $\Phi_{\max}(I, \pi)$ gets its maximum if each λ_j takes its maximum value.

Thus, to obtain a permutation that is optimal with respect to the minmax criterion, it is enough to solve the problem of minimizing function $\Phi_{\max}(I^+, \pi)$, where I^+ is the scenario with $\lambda_j = \lambda_j^+$ for all $j = 1, \ldots, n$. So, the permutation may be constructed by Algorithm 1.

The following statement characterizes the quality of solutions obtained according to the minmax approach.

Theorem 2 *Let π^+ be an optimal permutation for the scenario I^+. Then for any scenario I^0 the bound*

$$\Phi_{\max}(I^0, \pi^+) - \Phi_{\max}(I^0, \pi^*(I^0)) \leq$$

$$\max\{\lambda_j^+ - \lambda_j^- |\ j = 1, ..., n\}$$

holds and this bound is tight.

Proof. Let the scenario I^0 be defined by vector $\lambda^0 = (\lambda_1^0, \ldots, \lambda_n^0)$. For simplicity, denote permutation $\pi^*(I^0)$ by π^0. Without loss of generality, suppose that the jobs are numbered in accordance with permutation π^0. Let job l be a critical job in permutation π^0 for the scenario I^+, i.e., $\Phi_{\max}(I^+, \pi^0) = \varphi(\sum_{j=1}^{l} p_j) + \lambda_l^+$. Then $\varphi(\sum_{j=1}^{l} p_j) + \lambda_l^+ \geq \Phi_{\max}(I^+, \pi^+)$.

For I^0 we have $\Phi_{\max}(I^0, \pi^0) \geq \varphi(\sum_{j=1}^{l} p_j) + \lambda_l^0$. Therefore, $\Phi_{\max}(I^+, \pi^+) - \Phi_{\max}(I^0, \pi^0) \leq \lambda_l^+ - \lambda_l^0$.

Evidently, $\Phi_{\max}(I^+, \pi^+) \geq \Phi_{\max}(I^0, \pi^+)$. Thus, $\delta = \Phi_{\max}(I^0, \pi^+) - \Phi_{\max}(I^0, \pi^0) \leq \Phi_{\max}(I^+, \pi^+) - \Phi_{\max}(I^0, \pi^0) \leq \lambda_l^+ - \lambda_l^0 \leq \max_{1 \leq j \leq n}\{\lambda_j^+ - \lambda_j^-\}$.

Now we construct a simple example that shows the tightness of the bound. Given number $A > 1$, let $\varphi(t) = t$; $n = 2$; $p_1 = A$, $p_2 = 1$, $\lambda_1^- = 1+\varepsilon$, $\lambda_1^+ = A+1+\varepsilon$; $\lambda_2^- = 2, \lambda_2^+ = A+1$, where $\varepsilon > 0$ is an arbitrary small number. It is easy to see that $\pi^+ = (1, 2)$. Suppose that $\lambda_1^0 = 1 + \varepsilon, \lambda_2^0 = A + 1$. Then $\pi^0 = (2, 1)$, $\Phi_{\max}(I^0, \pi^+) = 2A + 2, \Phi_{\max}(I^0, \pi^0) = A + 2 + \varepsilon$. Thus, $\delta = A - \varepsilon$ and δ tends to A while ε tends to 0. ∎

Interesting to note that in case of the "optimistic" scenario I^- with $\lambda^- = (\lambda_1^-, \ldots, \lambda_n^-)$ we get the same bound.

Note 1 *Let π^- be an optimal permutation for the scenario I^-. Then for any scenario I^0 the bound*

$$\Phi_{\max}(I^0, \pi^-) - \Phi_{\max}(I^0, \pi^*(I^0)) \leq$$

$$\max\{\lambda_j^+ - \lambda_j^- \mid j = 1, ..., n\}$$

holds and this bound is tight.

The proof of this statement is similar to that of Theorem 2.

3.2 Minmax regret criterion

Define the minmax regret criterion for problem $1|prec; \lambda_j \in [\lambda_j^-, \lambda_j^+]|\Phi_{\max}$.

Definition 4 *For the problem $1|\, prec; \lambda_j \in [\lambda_j^-, \lambda_j^+]\,|\Phi_{\max}$, a permutation is called optimal with respect to the maximum regret if it minimizes the function*

$$\Delta(\mathcal{I}, \pi) = \max_{I \in \mathcal{I}}\{\Phi_{\max}(I, \pi) - \Phi_{\max}(I, \pi^*(I))\}. \quad (1)$$

The approach based on minimizing function (1) is known also as the robust deviation decision approach [5].

Denote by $\lambda' = (\lambda'_1, \ldots, \lambda'_n)$ the collection of the parameter values λ that defines scenario $I \in \mathcal{I}$ and rewrite the function $\Phi_{\max}(I, \pi) - \Phi_{\max}(I, \pi^*(I))$ from (1) in the following way:
$\Phi_{\max}(I, \pi) - \Phi_{\max}(I, \pi^*(I)) = \max_{j=1,\ldots,n}\{\varphi(C_j(\pi)) + \lambda'_j\} - \Phi_{\max}(I, \pi^*(I)) = \max_{j=1,\ldots,n}\{\varphi(C_j(\pi)) - (\Phi_{\max}(I, \pi^*(I)) - \lambda'_j)\}$.

Thus, $\Delta(\mathcal{I},\pi) =$

$$\max_{I\in\mathcal{I}}\{\max_{j=1,\dots,n}\{\varphi(C_j(\pi)) \ - \ (\Phi_{\max}(I,\pi^*(I)) \ - \ \lambda'_j)\}\} \ =$$

$$\max_{j=1,\dots,n}\{\max_{I\in\mathcal{I}}\{\varphi(C_j(\pi)) \ - \ (\Phi_{\max}(I,\pi^*(I')) \ - \ \lambda'_j)\}\} \ =$$

$$\max_{j=1,\dots,n}\{\varphi(C_j(\pi)) \ - \ \min_{I\in\mathcal{I}}\{\Phi_{\max}(I,\pi^*(I)) \ - \ \lambda'_j\}\}.$$

Definition 5 *For each $h \in \{1,\dots,n\}$ define the scenario $I_h \in \mathcal{I}$ by setting $\lambda_h = \lambda_h^+$ and $\lambda_j = \lambda_j^-$ for all $j \neq h$.*

Denote by π_h^* an optimal permutation for the scenario I_h. The following lemma shows the importance of scenarios I_h.

Lemma 2 *Given a scenario $I^0 \in \mathcal{I}$, let π^0 be an optimal permutation for I^0. Then for any $h \in \{1,\dots,n\}$ the following inequality is valid:*

$$\Phi_{\max}(I_h,\pi_h^*) \ - \ \lambda_h^+ \leq \Phi_{\max}(I^0,\pi^0) \ - \ \lambda_h^0, \tag{2}$$

where the collection $\lambda^0 = (\lambda_1^0,\dots,\lambda_n^0)$ defines scenario I^0.

Proof. Construct scenario I' replacing λ_j^0 with λ_j^- in scenario I^0 for all $j \neq h$. Evidently, $\Phi_{\max}(I',\pi^0) \leq \Phi_{\max}(I^0,\pi^0)$. Since $\Phi_{\max}(I_h,\pi_h^*) \leq \Phi_{\max}(I_h,\pi^0)$, to prove the validity of (2), it is enough to show that the inequality

$$\Phi_{\max}(I_h,\pi^0) \ - \ \lambda_h^+ \leq \Phi_{\max}(I',\pi^0) \ - \ \lambda_h^0$$

holds.

If h is not a critical job in π^0 for the scenario I_h, then $\Phi_{\max}(I_h,\pi^0) = \Phi_{\max}(I',\pi^0)$ and $\Phi_{\max}(I_h,\pi^0) - \lambda_h^+ \leq \Phi_{\max}(I',\pi^0) - \lambda_h^0$.

If h is a critical job, then $\Phi_{\max}(I_h,\pi^0) \ - \ \lambda_h^+ \ = (\varphi(C_h(\pi^0)) + \lambda_h^+) - \lambda_h^+ = (\varphi(C_h(\pi^0)) + \lambda_h^0) - \lambda_h^0 \leq \Phi_{\max}(I',\pi^0) \ - \ \lambda_h^0$. ∎

Corollary 4 *The maximum regret is of the form $\Delta(\mathcal{I},\pi) = max_{j=1,\dots,n}\{\varphi(C_j(\pi)) + d_j\}$, where $d_j = -(\Phi_{\max}(I_j,\pi_j^*) - \lambda_j^+)$.*

Proof. Lemma 2 implies that $\Delta(\mathcal{I},\pi) \ = \ max_{j=1,\dots,n}\{\varphi(C_j(\pi)) - min_{I\in\mathcal{I}}\{\Phi_{\max}(I,\pi^*(I)) - \lambda'_j\}\} \ = \ max_{j=1,\dots,n}\{\varphi(C_j(\pi)) - (\Phi_{\max}(I_j,\pi_j^*) - \lambda_j^+)\}$. ∎

Evidently, $\Delta(\mathcal{I},\pi)$ belongs to the class of maximum cost functions (with decomposable costs). Therefore, Lawler's algorithm may be used to minimize $\Delta(\mathcal{I},\pi)$. The following two-step algorithm constructs a permutation that gives the minimum to the maximum regret $\Delta(\mathcal{I},\pi)$.

Algorithm 2

1. *For each scenario* $I_j \in \mathcal{I},\ j = 1, \ldots, n$, *calculate the value* $\Phi_{\max}(I_j, \pi_j^*)$ *and set* $d_j = -\Phi_{\max}(I_j, \pi_j^*) + \lambda_j^+$.
2. *Apply Algorithm 1 to problem* $1|prec|\Delta_{\max}$ *with cost function* $\Delta_j(t) = \varphi(t) + d_j$ *to obtain an optimal permutation* π^* *and the minimum value* $\Delta^* = \Delta_{\max}(\pi^*)$.

The validity of Algorithm 2 is immediate. Using Corollary 4 and Definition 4, π^* is an optimal permutation and $\Delta^* = \Delta(\mathcal{I}, \pi^*)$ is the minimum value for the minmax regret criterion.

Below we will show that the running time of Step 1 of Algorithm 2 is $O(n^2)$ if we use a special procedure to implement this step. Thus, the running time of Algorithm 2 is $O(n^2)$.

Note that the idea of using scenarios I_h in the minmax regret approach is due to Averbakh [1].

Consider the following interesting special case.

Definition 6 *A permutation* π *is called globally optimal for problem* $1|\,prec; \lambda_j \in [\lambda_j^-, \lambda_j^+]\,|\Phi_{\max}$ *if* π *is optimal for all scenarios* $I \in \mathcal{I}$ *simultaneously.*

We get immediately two characterizations for the global optimality.

Theorem 3 *A globally optimal permutation* π *is obtained if Algorithm 2 returns the value* $\Delta^* = 0$. *If* $\Delta^* > 0$ *then no globally optimal permutation exists. This also means that* π *is globally optimal if and only if* π *is optimal for all instances* I_h.

Consider a simple example illustrating the notion of global optimality.

Set $\varphi(t) = t;\ n = 3;\ p_1 = 1,\ p_2 = 3,\ p_3 = 0;\ \lambda_1^- = \lambda_1^+ = 1,\ \lambda_2^- = 5,\ \lambda_2^+ = 7,\ \lambda_3^- = 4,\ \lambda_3^+ = 8$. Graph G has a unique arc $(1, 2)$.

For this instance we have only three feasible permutations: $(1, 2, 3)$, $(1, 3, 2)$ and $(3, 1, 2)$. Algorithm 2 gives us permutation $\pi = (3, 1, 2)$ and it is a globally optimal permutation (since $\Delta(I, \pi) = 0$). Interesting to note that the minmax approach produces the same permutation, whereas the minmin approach generates permutation $\pi^- = (1, 2, 3)$, which is not globally optimal (e.g., we have $\Phi_{\max}(I^+, \pi) = 11$ and $\Phi_{\max}(I^+, \pi^-) = 12$). There is one more globally optimal permutation $(1, 3, 2)$ but this permutation cannot be constructed with the listed approaches.

Consider a less simple example, which shows that a globally optimal permutation may exist in quite complicated situations.

Set $\varphi(t) = t;\ n = 8$. Values $p_j,\ \lambda_j^-$ and λ_j^+ are presented in Table 1.

Jobs	1	2	3	4	5	6	7	8
p_j	1	32	6	20	27	29	29	4
λ_j^-	43	32	47	33	46	54	44	58
λ_j^+	54	41	57	39	56	68	57	70

Table 1: Values $p_j,\ \lambda_j^-$ and λ_j^+

Graph G is presented by the set of its arcs: $(4,5)$, $(3,6)$, $(6,7)$, $(6,2)$, $(5,2)$, $(8,5)$. Here we do not list the transitive arcs of the graph.

We have $\pi = (8,3,6,7,4,5,1,2)$ as a globally optimal permutation.

Now, we are coming back to the justification of the time complexity of Algorithm 2. Given problem $1|prec;\ \lambda_j \in [\lambda_j^-, \lambda_j^+]\,|\Phi_{\max}$, we describe an algorithm for calculating the values $\Phi_{\max}(I_h, \pi_h^*)$ for all n scenarios $I_h \in \mathcal{I},\ h = 1, \ldots, n$, in $O(n^2)$ time.

Algorithm 3

1. *Construct the scenario $I^- \in \mathcal{I}$ by setting $\lambda_j = \lambda_j^-$ for all $j = 1, \ldots, n$.*
   Apply Algorithm 1 to construct an optimal permutation π^- for the scenario I^-.
   Compute the value $\Phi_{\max}(I^-, \pi^-)$.
   Renumber the jobs according to permutation π^- (i.e., $\pi^- = (1, \ldots, n)$ after the renumbering).
   Compute $C_j(\pi^-)$ for all $j = 1, \ldots, n$.
   Set $h = n$.

2. *Construct the scenario I_h replacing λ_h^- with λ_h^+ in I^-.*
   Set $v = h - 1,\ \pi_h^* = \pi^-$ and $J_h = (h)$.

3. *If $C_h(\pi^-) + \lambda_h^+ \le \Phi_{\max}(I^-, \pi^-)$, then set*
   $\Phi_{\max}(I_h, \pi_h^*) = \Phi_{\max}(I^-, \pi^-)$ and go to Step 7.

4. *If $v = 0$ or $\lambda_h^+ \le \lambda_v^-$, then go to Step 6, otherwise, go to Step 5.*

5. *If job v is a predecessor of job h, then set $J_h = (v, J_h)$. Otherwise, modify permutation π_h^* exchanging positions of job v and the subper-*

mutation J_h.
Set $v = v - 1$ *and go to Step 4.*

6. *Calculate* $\Phi_{\max}(I_h, \pi_h^*)$ *for the current permutation* π_h^*.

7. *Set* $h = h - 1$. *If* $h \geq 1$, *then go to Step 2, otherwise, the process of calculating the values* $\Phi_{\max}(I_h, \pi_h^*)$ *has been completed.*

Theorem 4 *Algorithm 3 finds permutations* π_h^* *correctly,* $h = 1, \ldots, n$, *and its running time does not exceed* $O(n^2)$.

Proof. According to Corollary 3, none of the jobs in permutation π^- has a weak improvement. After transforming scenario I^- into I_h, job h is the only job in π^- that may have a weak improvement.

In Step 5 of Algorithm 3, we eliminate weak improvements if they exist. Indeed, if job h has a weak improvement in π^-, then there is job l, $l < h$ such that $\lambda_l^- < \lambda_h^+$ and $A_G(l) \cap \{\pi(l+1), \ldots, \pi(h)\} = \emptyset$. Thus, if $\lambda_h^+ \leq \lambda_v^-$ and $v \notin B_G(h)$ in Step 5, then h has no weak improvements in the current permutation π_h^*. Otherwise, we would have $\lambda_l^- < \lambda_v^-$ and $v \notin A_G(l)$, i.e., job v would have a weak improvement in π^-.

Thus, if h really has a weak improvement in the current permutation π_h^*, then there are only two possibilities for job v. Either $v \to h$ or $\lambda_v^- < \lambda_h^+$. In the first case, we include v into the current subpermutation J_h, in the latter case, we interchange the positions of v and J_h and obtain a new feasible permutation π_h^* with the value $\Phi_{\max}(I_h, \pi_h^*)$, which is not greater than that for the previous permutation π_h^* (see Lemma 1). In the new permutation none of the jobs (excluding possibly job h) has a weak improvement.

If we have a situation with $v = 0$ or $\lambda_h^+ \leq \lambda_v^-$, then in the current permutation π_h^*, job h has no weak improvements. In view of Corollary 2, the final permutation π_h^* is optimal for the scenario I_h.

If the inequality $C_h(\pi^-) + \lambda_h^+ \leq \Phi_{\max}(I^-, \pi^-)$ holds in Step 3, then π^- is an optimal permutation for the scenario I_h. Indeed, if $\lambda_h^+ > \lambda_h^-$, then the validity of the above inequality means that there exists a critical job $j \neq h$ in π^- and the optimality of π^- follows from Corollary 2.

The running time of Algorithm 3 is $O(n^2)$. Really, Step 1 takes $O(n^2)$ time. For each h, each run of steps 2–5 and 7 takes $O(1)$ time (for each pair v and h checking the validity (or invalidity) of the relation $v \to h$ takes $O(1)$ time if we use the representation of graph G by its adjacency matrix). Each of the steps 2–5 and 7 repeats at most n times. Step 6 takes $O(n)$ time and for each h it runs at most once. Thus, the overall complexity of steps 2–7 does not exceed $O(n^2)$. ∎

4 Conclusion

Our two main results related to the well-known min-max cost algorithm of Lawler [2] are necessary and sufficient conditions for the sequence optimality and an $O(n^2)$ algorithm for constructing a schedule that minimizes the maximal regret criterion for the problem under uncertainty when the cost functions are decomposable.

Possible directions for further research include both considering more general cost functions (compared to decomposable functions) and addressing situations with more than one uncertain parameter.

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