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Equitable total coloring of cubic graphs is NP-complete*

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Abstract

A total coloring is equitable if the number of elements colored with each color differs by at most one, and the least integer for which a graph has such a coloring is called its equitable total chromatic number. It is conjectured that the equitable total chromatic number of a graph is at most $\Delta + 2$, and this has already been proved for cubic graphs. Therefore, the equitable total chromatic number of a cubic graph is either 4 or 5. In this work we prove that the problem of deciding whether it is 4 is NP-complete for bipartite cubic graphs.

Furthermore, we establish for one infinite family of Type 1 cubic graphs that they all have equitable total chromatic number 4.

Keywords: total coloring, equitable total coloring, cubic graphs

1 Introduction

The Total Coloring Conjecture states that the total chromatic number of any graph is at most $\Delta + 2$, where Δ is the maximum degree of the graph [1]. This conjecture has been proved for cubic graphs, so the total chromatic number of a cubic graph is either 4 (in which case the graph is called *Type 1*) or 5 (*Type 2*) [8, 11], see also [5] for a recent concise proof. In 1989, Sánchez-Arroyo [9] showed that the problem of deciding whether the total chromatic number of a bipartite cubic graph is 4 is NP-complete.

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The equitable total coloring requires further that the cardinalities of any two color classes differ by at most 1. Similarly to the situation with total colorings, it was conjectured that the equitable total chromatic number of any graph is at most $\Delta + 2$, and this conjecture was proved for cubic graphs in the same work [13]. We show in Section 3 that it is NP-complete to decide whether the equitable total chromatic number of a bipartite cubic graph is equal to 4.

Since, by definition, the equitable total chromatic number of a graph cannot be smaller than its total chromatic number, we get that if a cubic graph has no 4-total-coloring then not only does it have a 5-total-coloring, but also an equitable one. On the other hand, the equitable total chromatic number of a Type 1 cubic graph could be either 4 or 5.

Recently, a construction that allows to obtain infinitely Type 1 cubic graphs of equitable total chromatic number 5 has been defined [4]. In the same paper, it was shown that all but one generalized Petersen graphs $G(n, 2)$ and $G(n, 1)$ have equitable total chromatic number 4. The exception is $G(5, 1)$ which is Type 2.

In Section 4 we show that all Goldberg graphs [6] have equitable total chromatic number 4. These graphs were already partly determined to be Type 1 [3] but the known 4-total-colorings were not equitable.

2 Definitions

A *semi-graph* is a triple $G = (V(G), E(G), S(G))$ where $V(G)$ is the set of vertices of G, $E(G)$ is a set of edges having two distinct endpoints in $V(G)$, and $S(G)$ is a set of *semi-edges* having one endpoint in $V(G)$. When there is no chance of ambiguity, we simply write V, E or S.

We write edges having endpoints v and w shortly as vw. When vertex v is an endpoint of $e \in E \cup S$ we say that v and e are *incident*. Two elements of $E \cup S$ incident to the same vertex, respectively two vertices incident to the same edge, are called *adjacent*.

A *graph* G is a semi-graph with an empty set of semi-edges. In that case we can write $G = (V, E)$. Given a semi-graph $G = (V, E, S)$, we call the graph (V, E) the *underlying graph of* G.

All definitions given below for semi-graphs, that do not require the existence of semi-edges, are also valid for graphs.

Let $G = (V, E, S)$ be a semi-graph. The *degree* $d(v)$ of a vertex v of G is the number of elements of $E \cup S$ that are incident to v. We say that G is *d-regular* if the degree of each vertex is equal to d. In this paper we are mainly interested in 3-regular graphs and semi-graphs, also called respectively *cubic graphs* and *cubic semi-graphs*. A graph whose vertices have degree at most 3 is called *subcubic graph*. Given a graph G of maximum degree 3, the semi-graph obtained from G by adding $3 - d(v)$ semi-edges with endpoint v, for each vertex v of G, is called *the cubic semi-graph generated by* G.

For $k \in \mathbb{N}$, a *k-vertex-coloring* of G is a map $C^V \colon V \to \{1, 2, ..., k\}$, such that $C^V(x) \neq C^V(y)$ whenever x and y are two adjacent vertices. The *chromatic number* of G, denoted by $\chi(G)$, is the least k for which G has a k-vertex-coloring.

Similarly, a *k-edge-coloring* of G is a map $C^E \colon E \cup S \to \{1, 2, ..., k\}$, such that $C^E(e) \neq C^E(f)$ whenever e and f are adjacent elements of $E \cup S$. The *chromatic index* of G, denoted by $\chi'(G)$, is the least k for which G has a k-edge-coloring. By Vizing's theorem [12] we have that $\chi'(G)$ is equal to either $\Delta(G)$ or $\Delta(G) + 1$, where $\Delta(G)$ is the maximum degree of the vertices of G. If $\chi'(G) = \Delta(G)$, then G is said to be *Class 1*, otherwise G is said to be *Class 2*.

A *k-total-coloring* of G is a map $C^T\colon V \cup E \cup S \to \{1, 2, ..., k\}$, such that

(a) $C^T|_V$ is a vertex-coloring,

(b) $C^T|_{E\cup S}$ is an edge-coloring,

(c) $C^T(e) \neq C^T(v)$ whenever $e \in E \cup S$, $v \in V$, and e is incident to v.

The *total chromatic number* of G, denoted by $\chi''(G)$, is the least k for which G has a k-total-coloring. Clearly $\chi''(G) \geq \Delta(G)+1$. The Total Coloring Conjecture [1] claims that $\chi''(G) \leq \Delta+2$ and it has been proved for cubic graphs [8, 11]. A cubic graph is said to be Type 1 or Type 2, according to the fact that its total chromatic number is 4 or 5, respectively.

A *proper partial k-coloring* of G is an assignment of at most k colors to some elements of G such that adjacent or incident elements have different colors.

A k-total-coloring is *equitable* if the cardinalities of any two color classes differ by at most one. The least k for which G has an equitable k-total-coloring is the *equitable total chromatic number* of G, denoted by $\chi''_e(G)$. We remind the reader of the conjecture that $\chi''_e(G) \leq \Delta + 2$ for any graph G, and that this conjecture was proved for cubic graphs in the same work [13].

The next proposition, which is about equitable $k+1$-total-colorings of k-regular graphs, will be used in Sections 3 and 4.

Proposition 1 *Let G be a k-regular graph of Type 1 and let C^T be a $k+1$-total-coloring of G. The following three statements are equivalent.*

(a) *C^T is equitable,*

(b) *$C^T|_E$ is equitable (the numbers of edges in each color class differ by at most one)*

(c) *$C^T|_V$ is almost equitable (the numbers of vertices in each color class differ by at most two)*

Proof. For $1 \leq i \leq k+1$ let us denote by n_i, resp. m_i, the number of vertices, resp. edges, that are of color i in C^T. Then, by definition, the total number of elements in the color class i, denoted by t_i, is such that $t_i = m_i + n_i$ **(1)**. Furthermore, as G is k-regular, for each color i and each vertex v, the color i is given by C^T either to v or to an edge incident to v: so the number n of vertices of G verifies $n = 2m_i + n_i$, for each $1 \leq i \leq k+1$ **(2)**. Consider now any two colors i and j, $1 \leq i, j \leq k+1$. By **(2)** we get that $n_j - n_i = 2(m_i - m_j)$ and then by **(1)** we get that $t_i - t_j = m_j - m_i = \frac{n_i - n_j}{2}$. The equivalence of the statements in the proposition follows immediately from these equalities. □

The previous result helps us verify whether a 4-total-coloring is equitable — instead of considering the whole 4-total-coloring, we can just look at its restriction to the edges, or to the vertices, of the graph.

A *triangle* is a graph consisting of a cycle of length 3 (or equivalently a complete graph on three vertices) and a *square* is a graph consisting of a chordless cycle of length 4. We denote by $K_{3,3}$ the unique complete bipartite cubic graph.

Next we present a useful lemma that can be easily checked.

Lemma 1 *In every 4-total-coloring of the cubic semi-graph generated by a square, semi-edges incident to adjacent vertices get distinct colors.*

Let $G = (V, E)$ be a graph. The *girth of* G is the minimum length of a cycle contained in G, or if G has no cycle, it is defined to be infinity. A *snark* is a cyclically 4-edge-connected (and so of girth at least 4) Class 2 cubic graph (see [10, 2] for more information about Type of snarks).

3 The problem is NP-complete

The proof in [9] of the NP-completeness of the problem of deciding whether a bipartite cubic graph has 4-total-coloring is based on a polynomial-time reduction from the NP-complete problem of deciding whether a 4-regular graph has a 4-edge-coloring [7]. In this section, we prove that the problem of deciding whether a bipartite cubic graph has an equitable 4-total-coloring is NP-complete too. Our proof uses a reduction from the same problem but we had to modify the gadget used in [9].

Theorem 1 *The problem of deciding whether a cubic bipartite graph has an equitable 4-total-coloring is NP-complete.*

Proof. [Proof of Theorem 1.] Since we can verify in polynomial time that a candidate coloring is a 4-total-coloring, and since we can verify if the coloring is equitable by counting the cardinality of each color class, the problem is in the class NP.

We use graphs K, H, D depicted on Figure 1. We call H^- the graph obtained from H by deleting one pendant edge and its pendant extremity.

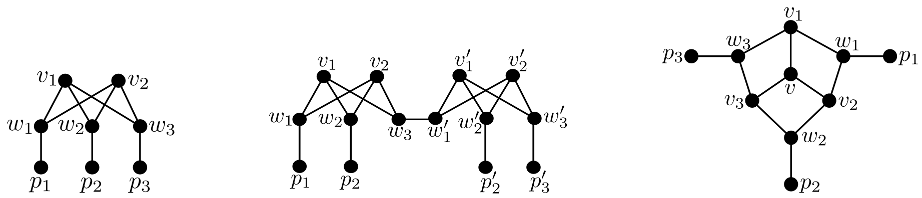

Figure 1: Subcubic graphs K, H, and D, respectively.

Lemma 2 (Sanchez-Arroyo [9]) *In each 4-total-coloring of K (resp. H) the three (resp. four) pendant edges of K (resp. H) receive the same color.*

Proof. Let C^T be a 4-total-coloring of K and let i be the color of v_1 in this coloring. Then none of w_1, w_2, w_3 can be colored i by C^T and so all these vertices should be incident to an edge colored i. We will show that these edges are the three pendant edges of K. Assume on the contrary that a non pendant edge is colored i, wlog say it is w_1v_2. Then the only way for color i to appear on w_2 is on the pendant edge incident to w_2. We then get a contradiction to Lemma 1 on the square induced by w_2, v_2, w_3, v_1. So the only way for color i to appear on w_1, w_2, w_3 is on their pendant edges. Remark that this also shows that v_1 and v_2 are colored the same in every 4-total-coloring of K. From our proof above, in every 4-total-coloring of H, all pendant edges of H receive the same color than the edge w_3w_1'. □

Definition 1 *Given a 4-total-coloring C^T of K (resp. H) we say it has characteristic i on K (resp. H) if i is the color given by C^T to the pendant edges of K (resp. H).*

Let us call R the graph obtained by removing the four pendant edges of the graph R^+ depicted on Figure 5. As shown by the black and white coloring of its vertices on the Figure 5, R is bipartite. Notice that R has two black vertices of degree 2, two white vertices of degree 2, and all other of degree 3.

Remark 1 *The set of edges of R is the union of the set of edges of disjoint copies of H^- and D.*

Construction of graph G^R from G Let G be a 4-regular graph. A graph G^R is built as follows. G^R contains a disjoint copy R_v of R, for each vertex v of G. Two copies of R are joined by an edge whenever the corresponding vertices are adjacent in G, so that there is a one-to-one correspondence between the set of edges of G and the set of edges of G^R that connect two copies of R. We call the edges connecting copies of R as *link edges* of G^R. In order to make G^R bipartite, we choose the extremities of the link edges as explained below. As G is 4-regular it is possible to orient its edges so that every vertex has both in- and out-degrees equal to 2 (Eulerian orientation). We add sequentially edges between copies of R using the following rule: if an edge of G is oriented from v to w put an edge between a white degree 2 vertex of R_v and a black degree 2 vertex of R_w. Any G^R obtained this way is cubic and bipartite. The construction of G^R can clearly be done in polynomial time in the order of G.

Definition 2 *One can associate to each vertex v of G a subgraph of G^R isomorphic to R^+ by adding to R_v the four link edges incident to R_v in G^R. We denote it by R_v^+.*

Claim 1 *If G^R is Type 1, then G is Class 1.*

Proof. [Proof of Claim 1] Assume that there exists a 4-total-coloring C^T of G^R, and let us consider the 4-total-coloring induced by C^T on R_v^+ for any vertex v of G. As any two of the four copies of H contained in R_v^+ have adjacent pendants, we get that C^T has distinct characteristics on these copies. In other words, we get that C^T gives four distinct colors to the link edges incident to R_v. So, giving to each edge vw of G the color given by C^T to the link edge connecting R_v and R_w we get a 4-edge-coloring of G. □

Claim 2 *If G^R is Type 1, then all 4-total-colorings of G^R are equitable.*

The next two lemmas will be used in the proof of Claim 2.

Lemma 3 *In any 4-total-coloring of H^- the number of edges in each color class is the same.*

Proof. Consider a 4-total-coloring of H^- of characteristic $i \in \{1, 2, 3, 4\}$. By Lemma 2 there are exactly four edges of H^- colored i: the three pendant edges and w_3w_1'. Furthermore v_1, v_2, v_1' and v_2' are also colored i. For each of these vertices the three incident edges are colored with the three colors distinct from i, say j, k, l. So there are exactly four edges of each color i, j, k, l among the edges in H^-. □

Lemma 4 *In any 4-total-coloring of D the number of edges in each color class is the same.*

Proof. Consider a 4-total-coloring of D and let i, j, k, l be the colors of resp. v, vv_1, vv_2, vv_3 in this coloring ($\{i, j, k, l\} = \{1, 2, 3, 4\}$). Then each of v_1, v_2 and v_3 should be incident to an edge of color i. By symmetry, we can assume that these edges are v_1w_1, v_2w_2 and v_3w_3. Then by Lemma 1, we get that w_3v_1, w_1v_2 and w_2v_3 are colored respectively k, l, j and then the coloring is forced to be as on Figure 2 and there are exactly three edges of each color. □

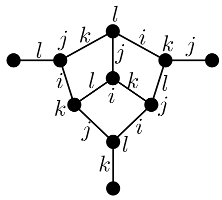

Figure 2: The graph D with a 4-total-coloring.

Proof. [Proof of Claim 2] Let C^T be a 4-total-coloring of G^R. Since there is a partition of the edges of G^R into the set of link edges and the sets of edges of disjoint copies of H^- and of D, it is enough to prove that in each set of the partition there are the same number of edges of each color. From the proof of Claim 1, we have that the colors of the link edges correspond to a 4-edge-coloring of G, so there are exactly $|V(G)|/2$ link edges having the same color i, for each $i = 1, 2, 3, 4$. Lemmas 3 and 4 conclude the proof. □

Claim 3 *If G is Class 1, then G^R is Type 1.*

The next two lemmas will be used in the proof of Claim 3.

Lemma 5 *Consider any proper partial 4-coloring C^P of H such that only w'_3, all pendant edges and all pendant vertices are colored, and:*

- *all the pendant edges have the same color, say i,*
- *p_1, p_2 have distinct colors, say resp. j and k (see Figure 3).*

This coloring C^P can be extended to the vertices $w_1, w_2, w_3, w'_1, w'_2$ and edge $w_3w'_1$ so that it is still proper and the colors of w_1, w_2, w_3 (resp. w'_1, w'_2, w'_3) are all distinct.

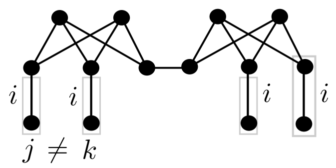

Figure 3: The framed elements are already colored by the proper partial 4-coloring C^P.

Proof. Color $w_3w'_1$ with color i. Color w'_2 with any color different from i and from the colors of w'_3 and p'_2, color w'_1 with any color different from i and from the colors of w'_2 and w'_3 (as we have four available colors this is always possible). Let $l \in \{1, 2, 3, 4\} \setminus \{i, j, k\}$. If w'_1 is colored j, then give color j to w_2, color k to w_1, and color l to w_3. Else, color w_1, w_2, w_3 resp. with k, l, j. □

Lemma 6 *Consider a proper partial 4-coloring of the pendant edges of K and their extremities, such that all pendant edges are colored the same and w_1, w_2, w_3 are colored with the three other colors. This coloring may be extended to a 4-total-coloring of K.*

Proof. See Figure 4. □

As a corollary of Lemmas 5 and 6, we get that any partial coloring satisfying the conditions of Lemma 5 can be extended to a 4-total-coloring of H.

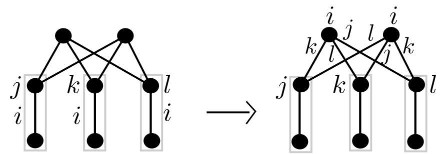

Figure 4: An extension of a proper partial 4-coloring of K satisfying the hypotheses of Lemma 6. The framed elements are already colored by a proper partial 4-coloring.

Proof. [Proof of Claim 3] Let C^E be a 4-edge-coloring of G. Starting from this coloring we will define a 4-total-coloring C^T of G^R. We define first the colors of the link edges of G^R: for every edge vw of G we give the color $C^E(vw)$ to the corresponding link edge E_{vw} of G^R: $C_1^T(E_{vw}) = C^E(vw)$.

Then, we give colors to the extremities of the link edges with any two available distinct colors. At this step the extension C_2^T of C_1^T is a proper partial 4-coloring of G^R that color, in each copy of R^+, all pendant edges and their extremities. For a vertex v of G, let the four link edges incident to R_v^+ be colored i, j, k, l by C_2^T as on Figure 5. Since C^E is a 4-edge-coloring of G, we have $\{i, j, k, l\} = \{1, 2, 3, 4\}$. We show in Figure 5 how this coloring can be extended to a proper coloring of all edges and vertices of R^+ that are not inside a $K_{2,3}$ of a copy of H. Doing this for every copy of R^+, we extend C_2^T to a proper partial 4-coloring C_3^T of G^R that colors the extremities of the link edges, and all other vertices and edges of G^R that are not inside a copy of H.

Noticing that the proper partial 4-coloring on Figure 5 is such that the conditions of Lemma 5 are verified for every copy of H in R^+, and since C_3^T colors every copy of R^+ as in Figure 5, we can apply Lemmas 5 and 6 in order to extend C_3^T into a 4-total-coloring C^T of G^R. □

From Claims 1, 2, and 3 we get that a 4-regular graph G has $\chi'(G) = 4$ if and only if the cubic graph G^R has $\chi''_e(G^R) = 4$. This ends the proof of Theorem 1. □

By Claim 1, whenever G has no 4-edge-coloring, the graph G^R is Type 2. Therefore, we do not prove that the problem of deciding whether a Type 1 cubic graph has equitable total chromatic number 4 is NP-complete.

Furthermore, since by Claim 2 whenever G^R has a 4-total-coloring then this 4-total-coloring is equitable, the problem of deciding whether G^R has an equitable 4-total-coloring is equivalent to the problem of deciding whether G^R is Type 1.

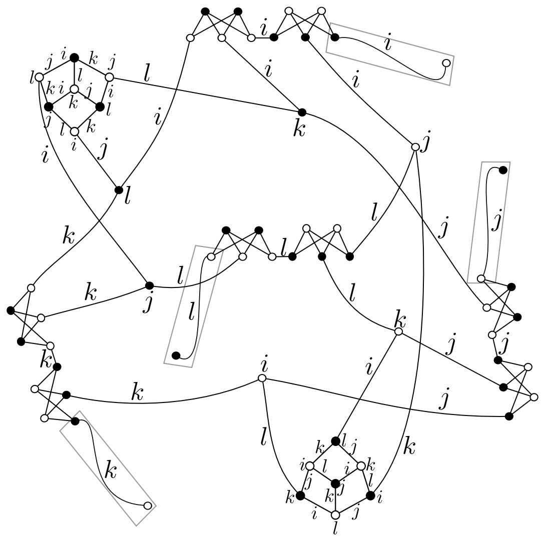

Figure 5: An extension of a proper partial 4-coloring of the framed elements of R^+.

4 Goldberg graphs

In this section, we determine that all Goldberg graphs have equitable total chromatic number 4. Let P be the Petersen graph, and let $Z_1, Z_2, ..., Z_n$ be n copies of the graph Z obtained by removing a path of size 3 of P. For every $n \geq 3$, the Goldberg graph GF_n is constructed by putting $Z_1, Z_2, ..., Z_n$ along a "cycle" as shown in Figure 6 for $n = 5$. For every odd $n \geq 3$, GF_n is a snark [6]. Campos et al. [3] determined a 4-total-coloring for each Goldberg snark, but it is not equitable. Note that Goldberg graphs have girth 5.

Theorem 2 *For every $n \geq 3$, the Goldberg graph GF_n has equitable total chromatic number* 4.

Proof. We notice that the colorings of $2Z$ and $3Z$ indicated on Figure 6 (notice that in this figure only edges get colors; the colors of the vertices can be deduced from them) have the property that all semi-edges of the same level have identical colors and any right and left vertices of the same level get distinct colors. Furthermore for each of these colorings there are the same number of vertices in each color class. So, they can be used sequentially in order to obtain, for any GF_n, a 4-total-coloring such that all color classes contain the same number of vertices. By Proposition 1 this 4-total-coloring is equitable. □

5 Conclusion

We conclude the paper by proposing two further questions, concerning the connection between the Type of a cubic graph and its equitable total chromatic number. Question 1 was motivated at the end of Section 3. All Type 2 cubic graphs have equitable total chromatic number 5, and since there exist examples of Type 1 cubic graphs with equitable total chromatic number 5, Question 2 is a natural question.

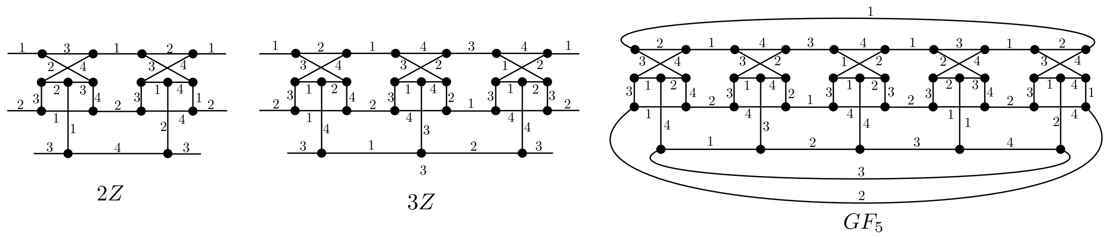

Figure 6: The semi-graphs $2Z$ and $3Z$ and the snark GF_5.

Question 1 *Is the problem of deciding whether a Type 1 cubic graph has equitable total chromatic number 4 NP-complete?*

Question 2 *Is the problem of deciding whether a cubic graph with equitable total chromatic number 5 is Type 1 NP-complete?*

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